CHAPTER 4

1. If x is the distance to the 14 kg end, then $10(10-x)=14x$, $100=24x$, and $x=4\frac{1}{6}$ m from the 14 kg end.
2. Since $8\cdot 10=80$ and $12\cdot 8=96$, the lever inclines toward the 12 kg weight.
3. Since a weight W of gold displaces a volume V_1 of fluid, a weight w_1 of gold will displace $\frac{w_1}{W}\cdot V_1$ of fluid. Similarly, a weight w_2 of silver displaces $\frac{w_2}{W}\cdot V_2$ of fluid. Thus the wreath, of total weight W, displaces the sum of these two amounts of fluid. That is, $V=\frac{w_1}{W}\cdot V_1+\frac{w_2}{W}\cdot V_2$. Since $W=w_1+w_2$, it follows that $(V-V_1)w_1=(V_2-V)w_2$,, and the desired result follows.
4. Lemma 1: $DA/DC=OA/OC$ by *Elements* VI–3. Therefore $DA/OA=DC/OC=(DC+DA)/(OC+OA)=AC/(CO+OA)$. Also, $DO^2=OA^2+DA^2$ by the Pythagorean Theorem.

   Lemma 2: $AD/DB=BD/DE=AC/CE=AB/BE=(AB+AC)/(CE+BE)=(AB+AC)/BC$. Therefore, $AD^2/BD^2=(AB+AC)^2/BC^2$. But $AD^2=AB^2-BD^2$. So $(AB^2-BD^2)/BD^2=(AB+AC)^2/BC^2$ and $AB^2/BD^2=1+(AB+AC)^2/BC^2$.
5. Set $r=1$, t_i and u_i as in the text, and P_i the perimeter of the ith circumscribed polygon. Then the first ten iterations of the algorithm give the following:

$t_1=.577350269$	$u_1=1.154700538$	$P_1=3.464101615$
$t_2=.267949192$	$u_2=1.03527618$	$P_2=3.21539031$
$t_3=.131652497$	$u_3=1.008628961$	$P_3=3.159659943$
$t_4=.065543462$	$u_4=1.002145671$	$P_4=3.146086215$
$t_5=.03273661$	$u_5=1.0005357$	$P_5=3.1427146$
$t_6=.016363922$	$u_6=1.00013388$	$P_6=3.141873049$
$t_7=.0081814134$	$u_7=1.000033467$	$P_7=3.141662746$
$t_8=.004090638249$	$u_8=1.000008367$	$P_8=3.141610175$
$t_9=.002045310568$	$u_9=1.000002092$	$P_9=3.141597032$
$t_{10}=.001022654214$	$u_{10}=1.000000523$	$P_{10}=3.141593746$

6. Let d be the diameter of the circle, t_i the length of one side of the regular inscribed polygon of $3\cdot 2^i$ sides, and u_i the length of the other leg of the right triangle formed from the diameter and the side of the polygon. Then

$$\frac{{t_{i+1}}^2}{d^2}=\frac{t_i^2}{t_i^2+(d+u_i)^2}$$

or

$$t_{i+1}=\frac{dt_i}{\sqrt{t_i^2+(d+u_i)^2}}\qquad u_{i+1}=\sqrt{d^2-{t_{i+1}}^2}.$$

If P_i is the perimeter of the ith inscribed polygon, then $\frac{P_i}{d}=\frac{3\cdot 2^i t_i}{d}$. So let $d=1$. Then $t_1=\frac{d}{2}=0.5$ and $u_1=\frac{\sqrt{3}d}{2}=0.8660254$. Then repeated use of the algorithm gives us:

$t_1=0.500000000$	$u_1=0.866025403$	$P_1=3.000000000$
$t_2=0.258819045$	$u_2=0.965925826$	$P_2=3.105828542$

$t_3 = 0.130526194$	$u_3 = 0.991444861$	$P_3 = 3.132628656$
$t_4 = 0.06540313$	$u_4 = 0.997858923$	$P_4 = 3.13935025$
$t_5 = 0.032719083$	$u_5 = 0.999464587$	$P_5 = 3.141031999$
$t_6 = 0.016361731$	$u_6 = 0.999866137$	$P_6 = 3.141452521$
$t_7 = 0.008181140$	$u_7 = 0.999966533$	$P_7 = 3.141557658$
$t_8 = 0.004090604$	$u_8 = 0.999991633$	$P_8 = 3.141583943$
$t_9 = 0.002045306$	$u_9 = 0.999997908$	$P_9 = 3.141590016$

7. Let the equation of the parabola be $y = -x^2+1$. Then the tangent line at $C = (1, 0)$ has the equation $y = -2x+2$. Let the point O have coordinates $(-a, 0)$. Then $MO = 2a+2$, $OP = -a^2+1$, $CA = 2$, $AO = -a+1$. So $MO : OP = (2a+2) : (1-a^2) = 2 : (1-a) = CA : AO$.

8.
   a. Draw line AO. Then $MS \cdot SQ = CA \cdot AS = AO^2 = OS^2 + AS^2 = OS^2 + SQ^2$.

   b. Since $HA = AC$, we have $HA : AS = MS : SQ = MS^2 : MS \cdot SQ = MS^2 : (OS^2+SQ^2) = MN^2 : (OP^2+QR^2)$. Since circles are to one another as the squares on their diameters, the latter ratio equals that of the circle with diameter MN to the sum of the circle with diameter OP and that with diameter QR.

   c. Since then $HA : AS =$ (circle in cylinder):(circle in sphere + circle in cone), it follows that the circle placed where it is is in equilibrium about A with the circle in the sphere together with the circle in the cone if the latter circles have their centers at H.

   d. Since the above result is true whatever line MN is taken, and since the circles make up the three solids involved, Archimedes can conclude that the cylinder placed where it is is in equilibrium about A with the sphere and cone together, if both of them are placed with their center of gravity at H. Since K is the center of gravity of the cylinder, it follows that $HA : AK =$ (cylinder):(sphere + cone).

   e. Since $HA = 2AK$, it follows that the cylinder is twice the sphere plus the cone AEF. But we know that the cylinder is three times the cone AEF. Therefore the cone AEF is twice the sphere. But the cone AEF is eight times the cone ABD, because each of the dimensions of the former are double that of the latter. Therefore, the sphere is four times the cone ABD.

9. Since $BOAPC$ is a parabola, we have $DA : AS = BD^2 : OS^2$, or $HA : AS = MS^2 : OS^2$. Thus $HA : AS =$ (circle in cylinder):(circle in paraboloid). Thus the circle in the cylinder, placed where it is, balances the circle in the paraboloid placed with its center of gravity at H. Since the same is true whatever cross section line MN is taken, Archimedes can conclude that the cylinder, placed where it is, balances the paraboloid, placed with its center of gravity at H. If we let K be the midpoint of AD, then K is the center of gravity of the cylinder. Thus $HA : AK =$ cylinder:paraboloid. But $HA = 2AK$. So the cylinder is double the paraboloid. But the cylinder is also triple the volume of the cone ABC. Therefore, the volume of the paraboloid is 3/2 the volume of the cone ABC which has the same base and same height.

10. Suppose the radius of the base of the cylinder is r and the height is h. The volume of the parallelepiped circumscribing the cylinder is $4r^2h$. To find the volume of the segment cut off by the plane, we note that the equation of the cutting plane is $z = (h/r)x$. Therefore,

the volume of the segment is

$$2\int_0^r \int_0^{\sqrt{r^2-x^2}} \frac{h}{r}x\,dy\,dx = 2\int_0^r \frac{h}{r}x\sqrt{r^2-x^2}\,dx = -\frac{2h}{3r}(r^2-x^2)^{3/2}\Big|_0^r = \frac{2h}{3r}r^3 = \frac{2}{3}hr^2.$$

This value is 1/6 of the volume of the parallelpiped, as desired.

11. Let the parabola be given by $y = a - bx^2$. Then the area A of the segment cut off by the x axis is given by

$$\begin{aligned} A &= 2\int_0^{\sqrt{a/b}} (a-bx^2)\,dx = 2\left(ax - \frac{1}{3}bx^3\right)\Big|_0^{\sqrt{a/b}} \\ &= 2a\sqrt{\frac{a}{b}} - \frac{2a}{3}\sqrt{\frac{a}{b}} = \frac{4a}{3}\sqrt{\frac{a}{b}}. \end{aligned}$$

Since the area of the inscribed triangle is $a\sqrt{\frac{a}{b}}$, the result is established.

12. Let the equation of the parabola be $y = x^2$, and let the straight line defining the segment be the line through the points $(-a, a^2)$ and (b, b^2). Thus the equation of this line is $(a-b)x + y = ab$, and its normal vector is $N = (a-b, 1)$. Also, since the midpoint of that line segment is $B = (\frac{b-a}{2}, \frac{b^2+a^2}{2})$, the x-coordinate of the vertex of the segment is $\frac{b-a}{2}$. If $S = (x, x^2)$ is an arbitrary point on the parabola, then the vector M from $(-a, a^2)$ to S is given by $(x+a, x^2-a^2)$. The perpendicular distance from S to the line is then the dot product of M with N, divided by the length of N. Since the length of N is a constant, to maximize the distance it is only necessary to maximize this dot product. The dot product is $(x+a, x^2-a^2)\cdot(a-b, 1) = ax-bx+a^2-ab+x^2-a^2 = ax-bx+x^2-ab$. The maximum of this function occurs when $a-b+2x = 0$, or when $x = \frac{b-a}{2}$. And, as we have already noted, the point on the parabola with that x-coordinate is the vertex of the segment. So the vertex is the point whose perpendicular distance to the base of the segment is the greatest.

13. Let r be the radius of the sphere. Then we know from calculus that the volume of the sphere is $V_S = \frac{4}{3}\pi r^3$ and the surface area of the sphere is $A_S = 4\pi r^2$. The volume of the cylinder whose base is a great circle in the sphere and whose height equals the diameter has volume is $V_C = \pi r^2(2r) = 2\pi r^3$, while the total surface area of the cylinder is $A_C = (2\pi r)(2r) + 2\pi r^2 = 6\pi r^2$. Therefore, $V_C = \frac{3}{2}V_S$ and $A_C = \frac{3}{2}A_S$, as desired.

14. Let A be the area bounded by one complete turn of the spiral and A_C the area of the circle. Then

$$A = \frac{1}{2}\int_0^2 \pi a^2\theta^2\,d\theta = \frac{1}{2}a^2\frac{\theta^3}{3}\Big|_0^{2\pi} = \frac{4}{3}\pi^3a^2 = \frac{1}{3}\pi(2\pi a)^2 = \frac{1}{3}A_C.$$

15. Suppose the cylinder P has diameter d and height h, and suppose the cylinder Q is constructed with the same volume but with its height and diameter both equal to f. It follows that $d^2 : f^2 = f : h$, or that $f^3 = d^2h$. It follows that one needs to construct the cube root of the quantity d^2h, and this can be done by finding two mean proportionals between 1 and d^2h, or, alternatively, two mean proportionals between d and h (where the first one will be the desired diameter f).

16. Since the focus of the parabola $y^2 = px$ is at $(\frac{p}{4}, 0)$, it follows that p must equal 100 and that the equation of the parabola must be $y^2 = 400x$. This implies that the vertical line perpendicular to the axis through, for example, the point 1 meter from the vertex would meet the parabola at the two points 20 meters above and below that point.
17. The focus of $y^2 = px$ is at $(\frac{p}{4}, 0)$. The length of the latus rectum is $2\sqrt{p\frac{p}{4}} = p$.
18. The equation of the ellipse can be rewritten as $\frac{p}{2a}x^2 - px + y^2 = 0$ or as $x^2 - 2ax + \frac{2a}{p}y^2 = 0$, or finally as
$$\frac{(x-a)^2}{a^2} + \frac{y^2}{pa/2} = 1.$$
Therefore the center of the ellipse is at $(a, 0)$ and $b^2 = \frac{pa}{2}$. The hyperbola can be treated analogously.
19. Let the parabola be $y^2 = px$ and the point $C = (x_0, y_0)$. Then the tangent line at C has slope $\frac{p}{2y_0}$, and the equation of the tangent line is $y = \frac{p}{2y_0}(x - x_0) + y_0$. If we set $y = 0$, we can solve this equation for x to get $x = -x_0$.
20. Begin with the ellipse given in the form $b^2x^2 + a^2y^2 = a^2b^2$. The slope of the tangent line at the point $C = (-x_0, y_0)$ is then $\frac{b^2x_0}{a^2y_0}$, and the equation of the tangent line is $y - y_0 = \frac{b^2x_0}{a^2y_0}(x + x_0)$. If we set $y = 0$, then the x-coordinate of the point where the tangent line intersects the axis is $x = -x_0 - \frac{a^2{y_0}^2}{b^2x_0}$. It is then straightforward to show that $AH : AG = (a - x) : (-a - x)$ is equal to $BH : BG = (a + x_0) : (-x_0 + a)$.
21. We can arrange the axes and origin so that the first conic is given by the equation $y^2 = ax + bx^2$. Then the second conic will be given by the general equation $Ax^2 + Bxy + y^2 + Dx + Ey + F = 0$, where we have chosen the coefficient of y^2 to be 1 for reasons that will soon be clear. To find the intersections, we can solve the first equation for y ($y = \pm\sqrt{ax^2 + bx}$)and substitute in the second. Since this will involve radicals in the new equation for x, we can put all of them on one side and then square both sides, resulting in a fourth degree equation for x. Such an equation has at most four real solutions. We need to show that these solutions for x lead to no more than four actual intersections. So rewrite the equation for the second conic as $y^2 = -(Ax^2 + Bxy + Dx + Ey + F)$ and subtract this from the equation for the first conic. This eliminates the y^2 term and gives $(Bx + E)y + (A + b)x^2 + (a + D)x + F = 0.$ an equation certainly equivalent to the fourth degree equation. Note that as long as $Bx + E$ is not 0, this equation can be solved for y in terms of x. In other words, each x that is a solution of the fourth degree equation corresponds to exactly one y, thus insuring that there are no more than four intersections. On the other hand, if one of the solutions of the fourth degree equation is $x = -E/B$, thus giving $Bx + E = 0$, then the equivalent equation reduces to a quadratic in x, which has at most two other solutions. In other words, although $x = -E/B$ may correspond to two actual intersections, one with a positive y value and one with a negative y value, there are then only two additional solutions to the original fourth degree equation, each corresponding to exactly one y value. (Note that if $B = E = 0$, then again there are only two solutions for x, but each may correspond to two different y-values.) Thus, there are at most four real intersection points to the two conics.
22. For two conics to be tangent to one another at a point (x_1, y_1), the fourth degree equation

mentioned in exercise 21 must have a double root at x_1. Thus, since there can be at most two double roots to a fourth degree equation, and since, by the same argument as in exercise 21, each double root can correspond to at most one value for y_1, there can be no more than two points at which two conics are tangent. (Note that if $x = -E/B$ is a double root of the fourth degree equation, it corresponds to two different y values and therefore does not give a tangent point.)

23.

a. Let the ellipse be given by the equation $b^2x^2 + a^2y^2 = a^2b^2$. Let P have coordinates (x_0, y_0). Then the slope of the tangent line at P is $-\frac{b^2x_0}{a^2y_0}$. Thus the equation of line DK is $y = -\frac{b^2x_0}{a^2y_0}x$. By solving this equation simultaneously with the equation of the ellipse, we get the coordinates of the point D as $(-\frac{ay_0}{b}, \frac{bx_0}{a})$. It follows that the slope of the tangent line at D is $\frac{b^2ay_0/b}{a^2bx_0/a} = \frac{y_0}{x_0}$, which is the slope of the diameter PG, as desired.

b. Given that the coordinates of P are (x_0, y_0), it follows that $\tan\theta = \frac{y_0}{x_0}$ as before. Similarly, since the coordinates of D are $(-\frac{ay_0}{b}, \frac{bx_0}{a})$, it follows that $\tan\alpha = -\frac{bx_0/a}{ay_0/b} = -\frac{b^2x_0}{a^2y_0}$.

c. Take an arbitrary point S in the plane with rectangular coordinates (x, y) and oblique coordinates (x', y'). By drawing lines from S parallel to the two original axes and to the two oblique axes, one can show that $x = x'\cos\theta - y'\cos(180 - \alpha) = x'\cos\theta + y'\cos\alpha$ and that $y = x'\sin\theta + y'\sin(180-\alpha) = x'\sin\theta + y'\sin\alpha$. If we replace x and y in the equation of the ellipse by their values in terms of x' and y', we get the equation specified in the problem, once we notice that $b^2\cos\theta\cos\alpha + a^2\sin\theta\sin\alpha = 0$, given the values for $\tan\theta$ and $\tan\alpha$ found in part b.

d. Let $y = (\tan\theta)x$ be the equation of the diameter PG. If we solve this equation simultaneously with the original equation for the ellipse, we find the coordinates of the point P to be

$$x = \frac{ab}{\sqrt{b^2 + a^2\tan^2\theta}}, \quad y = \frac{ab\tan\theta}{\sqrt{b^2 + a^2\tan^2\theta}}.$$

It follows that

$$a' = \sqrt{x^2 + y^2} = \frac{ab\sec\theta}{\sqrt{b^2 + a^2\tan^2\theta}}.$$

Similarly,

$$b' = \frac{ab\tan\alpha}{\sqrt{b^2 + a^2\tan^2\alpha}}.$$

Then

$$a'^2 = \frac{a^2b^2\sec^2\theta}{b^2 + a^2\tan^2\theta} = \frac{a^2b^2}{A} \qquad \text{or} \qquad A = \frac{a^2b^2}{a'^2}.$$

Similarly,

$$C = \frac{a^2b^2}{b'^2}.$$

If we substitute these values for A and C into the equation of the ellipse given in part (c), we get the equation $\frac{x'^2}{a'^2}+\frac{y'^2}{b'^2}=1$, or

$$y'^2 = b'^2\left(\frac{a'^2-x'^2}{a'^2}\right) = \frac{b'^2}{a'^2}(a'-x')(a'+x') = \frac{b'^2}{a'^2}x_1{}'x_2{}'$$

as desired.

e. Since $PF = a'\sin(\alpha-\theta)$ and $CD = b'$, we have $PF\times CD = a'b'\sin(\alpha-\theta) = \frac{a^2b^2\sin(\alpha-\theta)}{\sqrt{AC}}$. But since $b^2\cos\theta\cos\alpha + a^2\sin\theta\sin\alpha = 0$, it follows that $(b^2\cos\theta\cos\alpha + a^2\sin\theta\sin\alpha)^2 = 0$ and therefore that

$$AC = a^2b^2(\sin\alpha\cos\theta - \cos\alpha\sin\theta)^2 = a^2b^2\sin^2(\alpha-\theta).$$

Therefore,

$$PF\times CD = \frac{a^2b^2\sin(\alpha-\theta)}{ab\sin(\alpha-\theta)} = ab$$

as claimed.

24. For simplicity, we will show that the sum of the squares on the halves of two conjugate diameters are equal to the sum of the squares on the halves of the two axes. Using Fig. 4.32, let $P=(x_0,y_0)$ and let the equation of the ellipse be $b^2x^2+a^2y^2=a^2b^2$. To find the slope of the tangent line RP, we take differentials: $2b^2x\,dx + 2a^2y\,dy = 0$, so $dy/dx = -b^2x/a^2y$. Thus, the slope of RP is $-b^2x_0/a^2y_0$, and the equation of line DK is $y=-(b^2x_0/a^2y_0)x$. To find the coordinates of point D, we take the intersection of this line with the ellipse. That is, we substitute for y in the equation for the ellipse:

$$b^2x^2 + a^2\left(\frac{b^4x_0^2}{a^4y_0^2}\right)x^2 = a^2b^2.$$

Solving this for x and noting that the x-coordinate of D is negative, we get

$$x = -\frac{a^2y_0}{\sqrt{a^2y_0^2+b^2x_0^2}} = -\frac{a^2y_0}{ab} = -\frac{ay_0}{b} \qquad \text{and thus} \qquad y = \frac{bx_0}{a}.$$

Therefore, the square on CD is

$$\frac{a^2y_0^2}{b^2}+\frac{b^2x_0^2}{a^2}.$$

If we then add this to $x_0^2+y_0^2$, the square on CP, we get

$$\frac{a^2y_0^2}{b^2}+y_0^2+\frac{b^2x_0^2}{a^2}+x_0^2 = (a^2+b^2)\left(\frac{y_0^2}{b^2}+\frac{x_0^2}{a^2}\right) = a^2+b^2.$$

And, of course, the sum of the squares on AC and BC is also a^2+b^2.

25. By *Conics* II–8, if we pass a secant line through the hyperbola $xy = 1$ which goes through points M and N on that curve and points T and U on the y-axis and x-axis respectively (the asymptotes), then the segments TM and TN are equal. Thus, if we let M approach N, then the secant line approaches the tangent line at N and therefore the two line segments TN, NU between N and the asymptotes are equal. Therefore, the triangles TSN and NRU are congruent. If the coordinates of N are $(x_0, \frac{1}{x_0})$, then $TS = NR = \frac{1}{x_0}$, and $NS = x_0$. So the slope of the tangent line TNU is

$$\frac{TS}{SN} = -\frac{1/x_0}{x_0} = -\frac{1}{x_0^2}.$$

26. Suppose the coordinates of the point N are $(x, 0)$. Then, if the coordinates of G are $(g, 0)$, the relationship between the two points is given by $(g - x) : (a - x) = p : 2a$, so $g = \frac{p(a-x)}{2a} + x$. If $P = (x, y)$ and $Q = (x', y')$, then the square of the distance from G to Q is given by

$$y'^2 + \left(\frac{p(a-x)}{2a} + x - x'\right)^2 = x'\left(p - \frac{p}{2a}x\right) + \left(\frac{p(a-x)}{2a} + x - x'\right)^2.$$

If we take the derivative of the right side with respect to x' and set the result equal to 0, we find that $x' = x$. Thus the point P is the closest point on the curve to G. Also, the slope of the tangent line to the ellipse at P is given by $\frac{pa-px}{2ay}$. The slope of line PG is $-\frac{y}{p(a-x)/2a} = -\frac{2ay}{pa-px}$. Thus PG is perpendicular to the tangent line.

27. Unfortunately, the definition of similarity given is not quite accurate. It should say that two conic sections are similar when a ratio exists between the corresponding abscissas of the two curves such that the ratios of the corresponding ordinates to the abscissas are equal. In modern terms, two conics are similar if there is a similarity transformation between them, a function taking (x, y) to (kx, ky) for some k. Thus, given the parabolas $y^2 = px$ and $y'^2 = qx'$, consider the transformation $x \to (p/q)x', y \to (p/q)y'$. The transformed parabola is that $(p^2/q^2)y'^2 = (p^2/q)x'$, or $y'^2 = qx'$. Thus the similarity transformation has taken the first parabola to the second, and similarity is proved. To use the definition more directly, we can let x_1, x_2 be coordinates of two points on the x-axis of the first parabola and x_1', x_2' coordinates of two points on the x-axis of the second parabola, such that $x_1' : x_1 = x_2' : x_2 = q : p$. Let y_i and y_i' be the corresponding y-coordinates. We need to show that $y_i : x_i = y_i' : x_i'$ for $i = 1, 2$, or that $y_i' : y_i = x_i' : x_i$. We have ${y_i'}^2 : {y_i}^2 = qx_i' : px_i = q^2 : p^2$, so $y_i' : y_i = q : p = x_i' : x_i$ as desired.

28. According to proposition VI–12, two ellipses are similar if and only if $p_1 : a_1 = p_2 : a_2$, where p_i and a_i are the parameter and the length of the semi-major axis, respectively, of the i'th ellipse. But $p_i = 2b_i^2/a_i$, where b_i is the length of the semi-minor axis. So $p_1/a_1 = 2b_1^2/a_1^2$ and $p_2/a_2 = 2b_2/a_2^2$, so $b_1^2/a_1^2 = b_2/a_2^2$ and therefore $b_1/a_1 = b_2/a_2$. For hyperbolas, we have that two hyperbolas are similar if the ratios of the perpendicular distance from the vertex to the asymptote to the semi-major axis are equal. The proof is virtually identical.

29. We first need to show that $DB \cdot AC = pa/2 = b^2$. To do this, suppose $E = (x_0, y_0)$ and write the equation of the ellipse in the form $y^2 = \frac{p}{2a}x(2a - x)$, where B is the origin

of the coordinate system. We know from proposition I-34 that the x-coordinate of K is $(-t, 0)$, where $t = ax_0/(a - x_0)$. The equation of the tangent line is then

$$y = \frac{y_0}{x_0 + t}(x + t),$$

and we can then find the lengths of DB and AC. We find that

$$DB = \frac{ay_0}{2a - x_0} \qquad \text{and} AC = \frac{ay_0}{x_0}.$$

The product is then

$$\frac{a^2y_0^2}{x_0(2a - x_0(} = \frac{a^2y_0^2}{(2a/p)y_0^2} = \frac{pa^2}{2a} = \frac{pa}{2} = b^2.$$

Since $AF \cdot FB = b^2$, we have $AF \cdot FB = DB \cdot AC$, or $AC : AF = FB : DB$. Since the angles at A and B are right angles, triangles FAC and FBD are similar, so angle ACF is equal to angle BFD. But the sum of angles ACF and AFC is a right angle, so the sum of angles BFD and AFC is also a right angle. It follows that angle CFD is a right angle. By a similar argument, angle DGC is also a right angle.

30. Since angles CFD and DGC are right angles, the circle drawn with diameter CD will pass through both F and G. But then angles DCG and DFG will cut off the same arc of that circle, so they are equal. But also, we know that angle DFG is equal to angle ACF from exercise 29. Therefore, angle DCG is equal to angle ACF, as desired. By a similar argument, angle CDF is equal to angle BDG.

31. If EH is not perpendicular to to CD, then let HL be drawn from H perpendicular to CD. Since angle CDF equals angle BDG and angle HLD equals angle DBG (both are right), triangle DGB is similar to triangle LHD, so $GD : DH = BD : DL$. Also, triangles DGH and HFC are similar, having a common angle at H and right angles at G and F by exercise 29. So $GD : DH = FC : CH$. Also, since triangles AFC and LCH are similar by exercise 30, $FC : CH = AC : CL$. It follows that $BD : DL = AC : CL$ or $BD : AC = DL : CL$. But $BD : AC = BK : AK$, so also $DL : CL = BK : AK$. Now draw a line from E parallel to AC meeting AK at M. Given that KEC is a tangent line to the ellipse, it follows from proposition I–34 that $BK : KA = BM : AM$. But $BM : AM = DE : EC$. It follows that $DL : CL = BK : AK = BM : AM = DE : EC$, and this is impossible. Thus EH must be perpendicular to CD.

32. Since angles DGH and DEH are right angles, the circle drawn with diameter DH will pass through E and G. Therefore, since angles DHG and DEG cut off the same arc on this circle, they are equal. Similarly, using the circle through C, E, H, and F, angle CEF equals angle CHF. But angle CHF equals angle DHG. Therefore, angle CEF is equal to angle DEG as claimed.

33. If we apply a rectangle equal to one-fourth of the rectangle on the parameter N and the axis AB to the axis AB of a hyperbola that exceeds by a square figure, then the application results in two points F and G on the axis called the **foci** of the hyperbola. If the equation of the hyperbola is $y^2 = x(px + (p/2a)x)$, the coordinates of the two points

are $x = -a \pm \sqrt{a^2 + pa/2} = -a \pm \sqrt{a^2 + b^2}$. The analogue to III–48 is that if E is a point on the hyperbola and a tangent line is drawn at E, then the angles that the lines EF and EG make with the tangent line are equal. The analogue to III–52 is that the difference of the lengths of the two lines drawn in the previous sentence is equal to the axis of the hyperbola.

34. Let the parabola have the equation $y^2 = px$, with the focus at $(\frac{p}{4}, 0)$. Since the slope of the tangent line at the point $P = (x, y)$ is $\frac{p}{2y}$, it follows that the direction vector T of the tangent line can be written in the form $(2y, p)$. Similarly, the direction vector L of the line parallel to the axis can be written as $(1, 0)$ and the direction vector V of the line from P to the focus can be written as $(x - \frac{p}{4}, y)$. Then the cosine of the angle between T and L is $\frac{2y}{\sqrt{4y^2+p^2}}$. The cosine of the angle between T and V is given by

$$\frac{2y\left(x - \frac{p}{4}\right) + py}{\sqrt{4y^2 + p^2}\sqrt{\left(x - \frac{p}{4}\right)^2 + y^2}} = \frac{2xy + \frac{py}{2}}{\sqrt{4y^2 + p^2}\sqrt{\left(x + \frac{p}{4}\right)^2}} = \frac{2y\left(x + \frac{p}{4}\right)}{\sqrt{4y^2 + p^2}\left(x + \frac{p}{4}\right)} = \frac{2y}{\sqrt{4y^2 + p^2}}.$$

Since these two cosines are equal, so are the angles.

35. If the two parallel lines are $x = 0$ and $x = k$ and the perpendicular line is the x-axis, then the equation of the curve satisfying the problem is $y^2 = px(k - x)$ or $y^2 = kpx - px^2$. This is the equation of a conic section.

36. Since the square of the distance between a point and a line is a quadratic function of the coordinates x, y of the point, and since the same is true for the product of the distances to two separate lines, the equation defining the locus in the three-line problem will be a quadratic equation in x and y. Thus the locus will be a conic section, possibly a degenerate one.

37. Since $\angle BGA$ has been bisected, it follows from *Elements* VI–3 and similarity that $AG : BG = AD : BD = AE : EZ$. Therefore, triangle ADG is isosceles and $BG : EZ = AG : AE = 2 : 1$. Therefore, B lies on the hyperbola defined as the locus of points such that the ratio of their distances to the point G and to the line EZ is a constant greater than 1. To do the synthesis, we just have to construct a hyperbola defined in this manner. Then the intersection point B of that hyperbola with the arc AG enables the arc to be trisected.