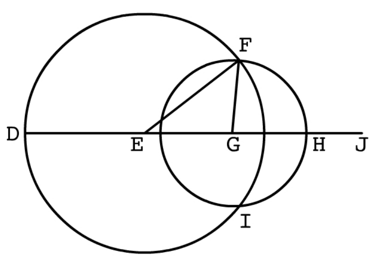
CHAPTER 3

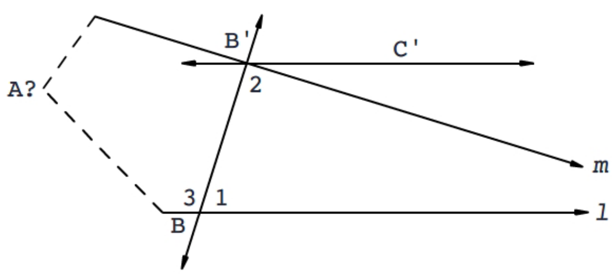
1. One way to do this is to use I-4. Namely, consider the isosceles triangle ABC with equal sides AB and BC also as a triangle CBA. Then the triangles ABC and CBA have two sides equal to two sides and the included angles also equal. Thus, by I-4, they are congruent. Therefore, angle BAC is equal to angle BCA, and the theorem is proved.
2. Put the point of the compass on the vertex V of the angle and swing equal arcs intersecting the two legs at A and B. Then place the compass at A and B respectively and swing equal arcs, intersecting at C. The line segment connecting V to C then bisects the angle. To show that this is correct, note that triangles VAC and VBC are congruent by SSS. Therefore, the two angles AVC and BVC are equal.
3. Let the lines AB and CD intersect at E. Then angles AEB and CED are both straight angles, angles equal to two right angles. If one subtracts the common angle CEB from each of these, the remaining angles AEC and BED are equal, and these are the vertical angles of the theorem.
4. Suppose the three lines have length a, b, and c, with $a \geq b \geq c$. On the straight line DH of length $a+b+c$, let the length of DF be a, the length of FG be b, and the length of GH be c. Then draw a circle centered on F with radius a and another circle centered at G with radius c. Let K be an intersection point of the two circles. Then connect FK and GK. Triangle FKG will then be the desired triangle. For $FK = FD$, and this has length a. Also FG has length b, while $GK = GH$ and this has length c. Note also that we must have $b+c > a$, for otherwise the two circles would not intersect. That $a+b > c$ and $a+c > b$ is obvious from how we have labeled the three lengths.

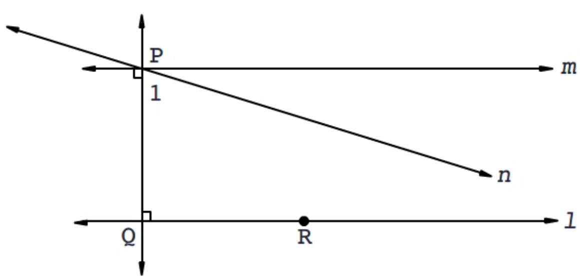


5. Suppose angle DCE is given, and we want to construct an angle equal to angle DCE at point A of line AB. Draw the line DE so that we now have a triangle DCE. (Here D and E are arbitrary points along the two arms of the given angle.) Then, by the result of exercise 4, construct a triangle AGF, with AG along line AB, where $AG = CE$, $AF = CD$, and $FG = DE$. By the side-side-side congruence theorem, triangle AGF is congruent to triangle CED. Therefore, angle FAG is equal to angle DCE, as desired.
6. Let ABC be the given triangle. Extend BC to D and draw CE parallel to AB. By I–29, angles BAC and ACE are equal, as are angles ABC and ECD. Therefore angle ACD equals the sum of the angles ABC and BAC. If we add angle ACB to each of these, we get that the sum of the three interior angles of the triangle is equal to the straight angle BCD. Because this latter angle equals two right angles, the theorem is proved.
7. Place the given rectangle $BEFG$ so that BE is in a straight line with AB. Extend FG to

H so that AH is parallel to BG. Connect HB and extend it until it meets the extension of FE at D. Through D draw DL parallel to FH and extend GB and HA so they meet DL in M and L respectively. Then HD is the diagonal of the rectangle $FDLH$ and so divides it into two equal triangles HFD and HLD. Because triangle BED is equal to triangle BMD and also triangle BGH is equal to triangle BAH, it follows that the remainders, namely rectangles $BEFG$ and $ABML$ are equal. Thus $ABML$ has been applied to AB and is equal to the given rectangle $BEFG$.

8. Because triangles ABN, ABC, and ANC are similar, we have $BN : AB = AB : BC$, so $AB^2 = BN \cdot BC$, and $NC : AC = AC : BC$, so $AC^2 = NC \cdot BC$. Therefore $AB^2 + AC^2 = BN \cdot BC + NC \cdot BC = (BN + NC) \cdot BC = BC^2$, and the theorem is proved.
9. In this proof, we shall refer to certain propositions in Euclid's Book I, all of which are proved before Euclid first uses Postulate 5. (That occurs in proposition 29.) First, assume Playfair's axiom. Suppose line t crosses lines m and l and that the sum of the two interior angles (angles 1 and 2 in the diagram) is less than two right angles. We know that the sum of angles 1 and 3 is equal to two right angles. Therefore $\angle 2 < \angle 3$. Now on line BB' and point B' construct line $B'C'$ such that $\angle C'B'B = \angle 3$ (Proposition 23). Therefore, line $B'C'$ is parallel to line l (Proposition 27). Therefore, by Playfair's axiom, line m is not parallel to line l. It therefore meets l. We must show that the two lines meet on the same side as C'. If the meeting point A is on the opposite side, then $\angle 2$ is an exterior angle to triangle ABB', yet it is smaller than $\angle 3$, one of the interior angles, contradicting proposition 16. We have therefore derived Euclid's postulate 5.

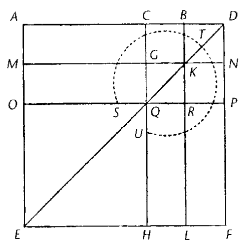




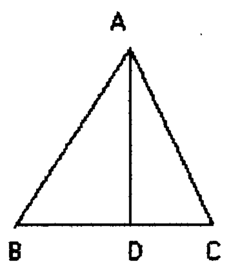
Second, assume Euclid's postulate 5. Let l be a given line and P a point outside the line. Construct the line t perpendicular to l through P (Proposition 12). Next, construct the line m perpendicular to line t at P (Proposition 11). Since the alternate interior angles formed by line t crossing lines m and l are both right and therefore are equal, it follows from Proposition 27 that m is parallel to l. Now suppose n is any other line through P. We will show that n meets l and is therefore not parallel to l. Let $\angle 1$ be the acute angle that n makes with t. Then the sum of angle 1 and angle PQR is less than two right angles. By postulate 5, the lines meet.

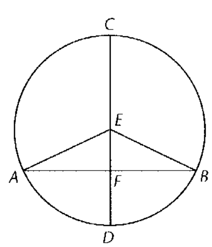
Note that in this proof, we have actually proved the equivalence of Euclid's Postulate 5 to the statement that given a line l and a point P not on l, there is at most one line through P which is parallel to l. The other part of Playfair's Axiom was proved (in the second part above) without use of postulate 5 and was not used at all in the first part.

10. One possibility for an algebraic translation: If the line has length a and is cut at a point with coordinate x, then $4ax + (a - x)^2 = (a + x)^2$. This is a valid identity. Here is a geometric diagram, with $AB = ME = a$ and $CB = BD = BK = KR = x$:



11. If ABC is the given acute-angled triangle and AD is perpendicular to BC, then the theorem states that the square on AC is less than the squares on CB and BA by twice the rectangle contained by CB and BD. If we label AC as b, BA as c, and CB as a, then $BD = c\cos B$. Thus the theorem can be translated algebraically into the form $b^2 = a^2 + c^2 - 2ac\cos B$, exactly the law of cosines in this case.

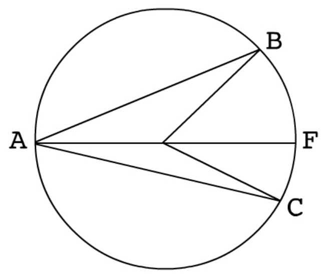




12. Suppose the diameter CD of a circle with center E bisects the chord AB at F. Then join EA and EB, forming triangle EAB. Triangles AEF and BEF are congruent by side-side-side (since $AE = BE$ are both radii of the circle and F bisects AB). Therefore angles EFA and EFB are equal. But the sum of those two angles is equal to two right angles. Hence each is a right angle, as desired. To prove the converse, use the same

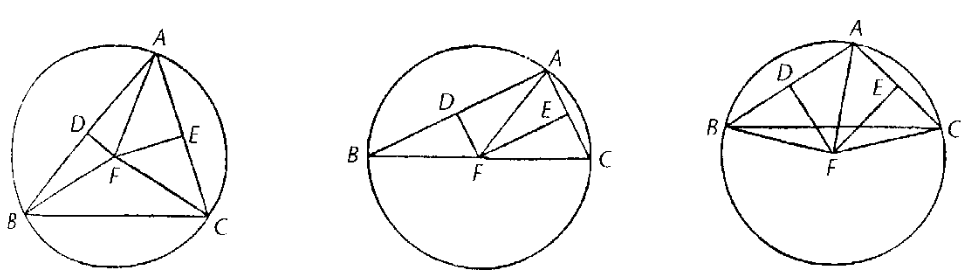
construction and note that triangle AEB is isosceles, so angle EAF is equal to angle EBF, while both angles EFA and EFB are right by hypothesis. It follows that triangles AEF and BEF are again congruent, this time by angle-angle-side. So $AF = BF$, and the diameter bisects the chord.

13. In the circle ABC, let the angle BEC be an angle at the center and the angle BAC be an angle at the circumference which cuts off the same arc BC. Connect $\angle EAB$. Similarly, $\angle FEC$ is double $\angle EAC$. Therefore the entire $\angle BEC$ is double the entire $\angle BAC$. Note that this argument holds as long as line EF is within $\angle BEC$. If it is not, an analogous argument by subtraction holds.



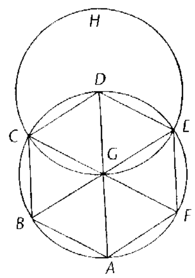
14. Let $\angle BAC$ be an angle cutting off the diameter BC of the circle. Connect A to the center E of the circle. Since $EB = EA$, it follows that $\angle EBA = \angle EAB$. Similarly, $\angle ECA = \angle EAC$. Therefore the sum of $\angle EBA$ and $\angle ECA$ is equal to $\angle BAC$. But the sum of all three angles equals two right angles. Therefore, twice $\angle BAC$ is equal to two right angles, and angle BAC is itself a right angle.

15. Let triangle ABC be given. Let D be the midpoint of AB and E the midpoint of AC. Draw a perpendicular at D to AB and a perpendicular at E to AC and let them meet at point F (which may be inside or outside the triangle, or on side BC). Assume first that F is inside the triangle, and connect FB, FA, and FC. Since $BD = BA$, triangles FDB and FDA are congruent by side-angle-side. Therefore $FB = FA$. Similarly, triangles FEA and FEC are congruent. So $FC = FA$. Therefore all three lines FA, FB, and FC are equal, and a circle can be drawn with center F and radius equal to FA. This circle will circumscribe the given triangle. Finally, note that the identical construction works if F is on line BC or if F is outside the triangle.



16. Let G be the center of the given circle and AGD a diameter. With center at D and radius DG, construct another circle. Let C and E be the two intersections of the two (equal) circles, and connect DC and DE. Then DE and DC are two sides of the desired regular hexagon. To find the other four sides, draw the diameters CGF and EGB. Then CB,

BA, AF, and FE are the other sides. To demonstrate that we have in fact constructed a regular hexagon, note that all the triangles whose bases are sides of the hexagon and whose other sides are radii are equilateral; thus all the sides of the hexagon are equal and all the angles of the hexagon are also equal.



17. In the circle, inscribe a side AC of an equilateral triangle and a side AB of an equilateral pentagon. Then arc BC is the difference between one-third and one-fifth of the circumference of the circle. That is, arc $BC = \frac{2}{15}$ of the circumference. Thus, if we bisect that arc at E, then lines BE and EC will each be a side of a regular 15-gon.

18. Let $a = s_1 b + r_1$, $b = s_2 r_1 + r_2$, …, $r_{k-1} = s_{k+1} r_k$. Then r_k divides r_{k-1} and therefore also $r_{k-2}, \ldots, b, a$. If there were a greater common divisor of a and b, it would divide r_1, r_2, …, r_k. Since it is impossible for a greater number to divide a smaller, we have shown that r_k is in fact the greatest common divisor of a and b.

19.
$$\begin{aligned} 963 &= 1 \cdot 657 + 306 \\ 657 &= 2 \cdot 306 + 45 \\ 306 &= 6 \cdot 45 + 36 \\ 45 &= 1 \cdot 36 + 9 \\ 36 &= 4 \cdot 9 + 0 \end{aligned}$$

Therefore, the greatest common divisor of 963 and 657 is 9.

$$\begin{aligned} 4001 &= 1 \cdot 2689 + 1312 \\ 2689 &= 2 \cdot 1312 + 65 \\ 1312 &= 20 \cdot 65 + 12 \\ 65 &= 5 \cdot 12 + 5 \\ 12 &= 2 \cdot 5 + 2 \\ 5 &= 2 \cdot 2 + 1 \end{aligned}$$

Therefore, the greatest common divisor of 4001 and 2689 is 1.

20.
$$\begin{array}{rclcrcrclcr} 46 &=& 7 \cdot 6 &+& 4 & \qquad & 23 &=& 7 \cdot 3 &+& 2 \\ 6 &=& 1 \cdot 4 &+& 2 & & 3 &=& 1 \cdot 2 &+& 1 \\ 4 &=& 2 \cdot 2 & & & & 2 &=& 2 \cdot 1 & & \end{array}$$

Note that the multiples 7, 1, 2 in the first example equal the multiples 7, 1, 2 in the second.

21.
$$\begin{array}{rclcl} 33 & = & 2\cdot 12 & + & 9 \\ 12 & = & 1\cdot 9 & + & 3 \\ 9 & = & 3\cdot 3 & & \end{array} \qquad \begin{array}{rclcl} 11 & = & 2\cdot 4 & + & 3 \\ 4 & = & 1\cdot 3 & + & 1 \\ 3 & = & 3\cdot 1 & & \end{array}$$

It follows that both ratios can be represented by the sequence $(2, 1, 3)$.

22. Since $1 - x = x^2$, we have

$$1 = 1 \cdot x + (1 - x) = 1 \cdot x + x^2$$

$$x = 1 \cdot x^2 + (x - x^2) = 1 \cdot x^2 + x(1 - x) = 1 \cdot x^2 + x^3$$

$$x^2 = 1 \cdot x^3 + (x^2 - x^3) = 1 \cdot x^3 + x^2(1 - x) = 1 \cdot x^3 + x^4$$

$$\dots$$

Thus $1 : x$ can be expressed in the form $(1, 1, 1, \dots)$.

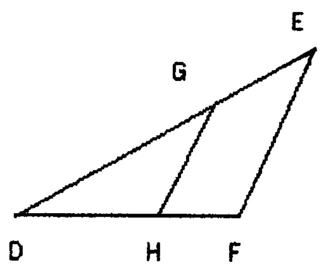
23. If d is the diagonal of a square of side s, then the first division gives $d = 1s + r$. To understand the next steps, it is probably easiest to set $s = 1$ and deal with the numerical values. Therefore, $d = \sqrt{2}$, and $r = \sqrt{2} - 1$. Our next division gives $s = 2r + t$, where $t = 3 - 2\sqrt{2}$. Geometrically, if we lay off s along the diagonal, then r is the remainder $d - s$. Then draw a square of side r with part of the side of the original square being its diagonal. Note that if we now lay off r along the diagonal of that square, the remainder is t. In other words, r is the difference between the diagonal of a square of side s and s, while t is the difference between the diagonal of a square of side r and r. It follows that if one performs the next division in the process, we will get the same relationship. That is, $r = 2t + u$, where now u is the difference between the diagonal of a square of side t and t. Thus, this process will continue indefinitely and the ratio $d : s$ and be expressed as $(1, 2, 2, 2, \dots)$.

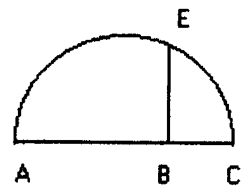
24. Since $a > b$, there is an integral multiple m of $a - b$ with $m(a - b) > c$. Let q be the first multiple of c that exceeds mb. Then $qc > mb \geq (q - 1)c$, or $qc - c \leq mb < qc$. Since $c < ma - mb$, it follows that $qc \leq mb + c < ma$. But also $qc > mb$. Thus we have a multiple (q) of c that is greater than a multiple (m) of b, while the same multiple (q) of c is not greater than the same multiple (m) of a. Thus by definition 7 of Book V, $c : b > c : a$.

25. Let $A : B = C : D = E : F$. We want to show that $A : B = (A + C + E) : (B + D + F)$. Take any equimultiples mA and $m(A+C+E)$ of the first and third and any equimultiples nB and $n(B+D+F)$ of the second and fourth. Since $m(A+C+E) = mA+mC+mE$, and since $n(B+D+F) = nB+nD+nF$, and since whenever $mA > nB$, we have $mC > nD$ and $mE > nF$, it follows that $mA > nB$ implies that $m(A + C + E) > n(B + D + F)$. Since a similar statement holds for equality and for "less than", the result follows from Eudoxus' definition. A modern proof would use the fact that $a_1 b_i = b_1 a_i$ for every i and then conclude that $a_1(b_1 + b_2 + \cdots + b_n) = b_1(a_1 + a_2 + \cdots + a_n)$.

26. Given that $a : b = c : d$, we want to show that $a : c = b : d$. So take any equimultiples ma, mb of a and b and also equimultiples nc, nd of c and d. Now $ma : mb = a : b = c : d = nc : nd$. Thus if $ma > nc$, then $mb > nd$; if $ma = nc$, then $mb = nd$; and if $ma < nc$, then $mb < nd$. Thus, by the definition of equal ratio, we have $a : c = b : d$.

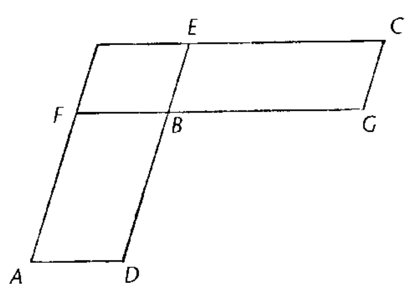
27. In the diagram, let $DG = 8$, $GE = 4$, and $DH = 6$. Connect GH and draw EF parallel to GH. Then HF is equal to the fourth proportional x.





28. Let $AB = 9$ and $BC = 5$. Draw a circle with AC as diameter and erect a perpendicular to AC at B, meeting the circle at E. Then BE is the desired length x.

29. Suppose the first of the equal and equiangular parallelograms has sides of length a and b while the second has sides of length c and d, each pair surrounding an angle equal to α. Since the area of a parallelogram is the product of the two sides with the sine of the included angle, we know that $ab \sin \alpha = cd \sin \alpha$. It follows that $ab = cd$ or that $a : c = d : b$, as desired. Conversely, if $a : c = d : b$ and the parallelograms are equiangular with angle α between each pair of given sides, then $ab = cd$ and $ab \sin \alpha = cd \sin \alpha$, so the parallelograms are equal. Euclid's proof is, of course, different from this modern one. Namely, if the two parallelograms are $P_1 = ADBF$ and $P_2 = BGCE$, with equal angles at B, Euclid places them so that FB and BG are in a straight line as are EB and BD. He then completes the third parallelogram $P_3 = FBEK$. Since $P_1 = P_2$, we have $P_1 : P_3 = P_2 : P_3$. But $P_1 : P_3 = DB : BE$, since BF is common, and $P_2 : P_3 = BG : BF$, since BE is common. Thus, $DB : BE = BG : BF$, the desired conclusion. The converse is proved by reversing the steps.



30. Suppose $a : b = f : g$ and suppose the numbers $c, d, \ldots, e$ are the numbers in continued proportion between a and b. Let $r, s, t, \ldots, u, v$ be the smallest numbers in the same ratio as $a, c, d, \ldots, e, b$. Then r, v are relatively prime and $r : v = a : b = f : g$. It follows that $f = mr$, $g = mv$ for some integer m and that the numbers $ms, mt, \ldots, mu$ are in the same ratio as the original set of numbers. Thus there are at least as many numbers in continued proportion between f and g as there were between a and b. Since the same argument works starting with f and g, it follows that there are exactly as many numbers in continued proportion between f and g as between a and b. Since there is no integer between n and $n + 1$, it follows that there cannot be a mean proportional between any pair of numbers in the ratio $(n + 1) : n$.

31. The number ab is the mean proportional between a^2 and b^2.

32. The numbers a^2b and ab^2 are the two mean proportionals between a^3 and b^3.

33. If a^2 measures b^2, then $b^2 = ma^2$ for some integer m. Since every prime number which divides b^2 must divide ma^2 and therefore must divide either m or a^2, it follows by counting primes that m must itself be a square. Thus $m = n^2$ and $b = na$, so a measures b. Conversely, if a measures b, then $b = na$ and $b^2 = n^2a^2$, so a^2 measures b^2.

34. Suppose m factors two different ways as a product of primes: $m = pqr \cdots s = p'q'r' \cdots s'$. Since p divides $pqr \cdots s$, it must also divide $p'q'r' \cdots s'$. By VII–30, p must divide one of the prime factors, say p'. But since both p and p' are prime, we must have $p = p'$. After canceling these two factors from their respective products, we can then repeat the argument to show that each prime factor on the left is equal to a prime factor on the right and conversely.

35. One standard modern proof is as follows. Assume there are only finitely many prime numbers p_1, p_2, p_3, ..., p_n. Let $N = p_1p_2p_3 \cdots p_n + 1$. There are then two possibilities. Either N is prime or N is divisible by a prime other than the given ones, since division by any of those leaves remainder 1. Both cases contradict the original hypothesis, which therefore cannot be true.

36. We keep adding powers of 2 until we get a prime. After $1+2+4+8+16+32+64 = 127$, the next sums are 255, 511, 1023, 2047, 4095, and 8191. The first five of these are not prime (note that $2047 = 23 \times 89$). But 8191 is prime (check by dividing by all primes less than $\sqrt{8191}$). So the next perfect number is $8191 \times 4096 = 33,550,336$.

37. Since BC is the side of a decagon, triangle EBC is a 36-72-72 triangle. Thus $\angle ECD = 108°$. Since CD, the side of a hexagon, is equal to the radius CE, it follows that triangle ECD is an isosceles triangle with base angles equal to 36°. Thus triangle EBD is a 36-72-72 triangle and is similar to triangle EBC. Therefore $BD : EC = EB : BC$ or $BD : CD = CD : BC$ and the point C divides the line segment BD in extreme and mean ratio.

38. By exercise 37, if we set d to be the length of the side of a decagon, we have $(1+d) : 1 = 1 : d$ or $d^2 + d - 1 = 0$. It follows that $d = \frac{\sqrt{5}-1}{2}$. The length p of the side of a pentagon is (p. 88) $p = \frac{1}{2}\sqrt{10 - 2\sqrt{5}}$. It is then straightforward to show that $p^2 = 1^2 + d^2$ as asserted.

39. We begin with a rectangle of sides x and y. We then lay off y along x, with the remainder being $x - y = 7$. Divide the rectangle with sides 7 and y in half and move one half to the bottom. We then add the square of side $\frac{7}{2}$ to get a square of side $y + \frac{7}{2} = x - \frac{7}{2}$ with area $18 + (\frac{7}{2})^2 = \frac{121}{4}$. Thus $y = \frac{11}{2} - \frac{7}{2} = 2$ and $x = \frac{11}{2} + \frac{7}{2} = 9$.

40. First, to solve the two equations, we set $x = a/y$ and substitute into $y^2 - \alpha x^2 = b$. After multiplying by y^2, we get the fourth degree equation $y^4 - \alpha a^2 = by^2$. This is quadratic in y^2, so we can solve to get

$$y^2 = \frac{b + \sqrt{b^2 + 4\alpha a^2}}{2}.$$

(Note that we only use the plus sign, since y^2 must be positive.) Then

$$y = \pm\sqrt{\frac{b + \sqrt{b^2 + 4\alpha a^2}}{2}}.$$

We can then solve for x:

$$x = \pm\sqrt{\frac{\sqrt{b^2 + 4\alpha a^2} - b}{2\alpha}}.$$

(Of course, since a is assumed positive, we take the positive solution for x when we have the positive one for y, as well as the negative solution for x with the negative one for y.) The asymptotes of the hyperbola $xy = a$ are the lines $x = 0$ and $y = 0$, and these are the axes of the hyperbola $y^2 - \alpha x^2 = b$. Thus the asymptotes of one hyperbola are the axes of the other. (Note that the problem as stated is incorrect; it should say that "one hyperbola has its axes on the asymptotes of the other.")