**Class 9: Tuesday March 24**

**TITLE** How Regular Perturbations on ODEs Can Go Wrong  
**CURRENT READING** Logan, Sections 2.1.2 and 2.1.3

**SUMMARY**  
This week we continue looking at regular perturbations in differential equations and stumble upon what can go wrong. We’ll be introduced to a method to still produce reasonable perturbation solutions called the Poincaré-Lindstedt method.

**RECALL**  
Given the IVP which models an object falling through a medium with air resistance proportional to current velocity squared

\[ m \frac{dv}{d\tau} = -av + bv^2, \quad v(0) = V_0 \]  

We can non-dimensionalize the model using the scalings

\[ y = \frac{v}{V_0}, \quad t = \frac{\tau}{m/a} \]

(2)

to produce

\[ \frac{dy}{dt} = -y + \epsilon y^2, \quad y(0) = 1 \text{ where } \epsilon = \frac{bV_0}{a} \ll 1 \]  

(3)

Similarly, given the following model for a nonlinear spring-mass oscillator

\[ m \frac{d^2y}{d\tau^2} = -ky - ay^3, \quad y(0) = A, \quad \frac{dy}{d\tau}(0) = 0 \]  

(4)

we can non-dimensionalize it using the scalings

\[ u = \frac{y}{A}, \quad t = \frac{\tau}{\sqrt{m/k}} \]

(5)

to produce

\[ \frac{d^2u}{dt^2} = -u - \epsilon u^3, \quad u(0) = 1, \quad u'(0) = 0 \text{ where } \epsilon = \frac{aA^2}{k} \ll 1 \]  

(6)

The IVP in (6) is known as Duffing’s Equation and has no known exact solution.  
If we assume a perturbation series solution of the form

\[ u(t) = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \ldots \]  

(7)

then we will produce a series of differential equations (with initial conditions) of various orders in epsilon...
EXAMPLE
Let’s show what the systems we get are:

The $O(1)$ equation is
\[
\frac{d^2 u_0}{dt^2} + u_0 = 0, \quad u_0(0) = 1, \quad u_0'(0) = 0
\] (8)

The $O(\epsilon)$ equation is
\[
\frac{d^2 u_1}{dt^2} + u_1 = -u_0^3, \quad u_1(0) = 0, \quad u_1'(0) = 0
\] (9)

The solution to the leading order IVP, the $O(1)$ term in (7) is
\[
u_0(t) = \cos(t)
\] (10)

which means that the $O(\epsilon)$ equation becomes
\[
\frac{d^2 u_1}{dt^2} + u_1 = -\cos^3(t), \quad u_1(0) = 0, \quad u_1'(0) = 0
\]

But using the common trigonometric identity $\cos(3t) = 4\cos^3(t) - 3\cos(t)$ the first-order equation (9) becomes
\[
\frac{d^2 u_1}{dt^2} + u_1 = -\frac{3}{4}\cos(t) - \frac{1}{4}\cos(3t), \quad u_1(0) = 0, \quad u_1'(0) = 0
\] (11)

which can be solved using the Method of Undetermined Coefficients (assume a solution of the form $A\cos(t) + B\sin(t) + C\cos(3t) + Dt\cos(t) + Et\sin(t)$ which produces the following solution (after applying the initial conditions)
\[
u_1(t) = \frac{1}{32}\cos(3t) - \frac{1}{32}\cos(t) - \frac{3}{8}t\sin(t)
\] (12)

EXAMPLE
We can confirm this above result.
Exercise

Confirm that the given functions in (10) and (12) are indeed the solution to the IVPs in (8) and (9), respectively.

Consider a graph of $u_0(t)$ and $u_0(t) + \epsilon u_1(t)$ plotted versus time for a typical value of $\epsilon = 0.1$ on the interval $0 \leq t \leq \frac{1}{\epsilon^2}$. What do you notice?

Therefore $\epsilon u_1(t)$ is NOT much less than $u_0(t)$ for all time. Can you explain what happens as $t$ gets larger and larger? Is it possible to estimate the value of $t$ where “trouble” begins?

Explain the significance of the Figure
The Poincaré-Lindstedt Method

In this technique the perturbation series is chosen to be

$$u(\tau) = u_0(\tau) + \epsilon u_1(\tau) + \epsilon^2 u_2(\tau) + \ldots$$  \hspace{1cm} (13)

where $\tau = \omega t$ and

$$\omega = \omega_0 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \ldots$$  \hspace{1cm} (14)

We can choose $\omega_0 = 1$ since it is the frequency of the solution given in (10) to the leading-order problem in Equation (9).

Using the new scalings given in (13) and (14) we can transform (6) into

$$\omega^2 \frac{d^2 u}{d\tau^2} = -u - \epsilon u^3, \quad u(0) = 1 \quad u'(0) = 0$$  \hspace{1cm} (15)

**EXAMPLE**

First let’s show how we get from (6) to (15)

and then we can show that the equations we get are:

The $\mathcal{O}(1)$ equations are

$$\frac{d^2 u_0}{d\tau^2} + u_0 = 0, \quad u_0(0) = 1, \quad u_0'(0) = 0$$  \hspace{1cm} (16)

The $\mathcal{O}(\epsilon)$ equations are

$$\frac{d^2 u_1}{d\tau^2} + u_1 = -2\omega_1 u_0'' - u_0^3, \quad u_1(0) = 0, \quad u_1'(0) = 0$$  \hspace{1cm} (17)
The solution to (16), \( \frac{d^2 u_0}{d\tau^2} + u_0 = 0, \quad u_0(0) = 1, \quad u_0'(0) = 0 \), is similar to the solution from (8) which turns out to be
\[
  u_0(\tau) = \cos(\tau)
\] (18)
which leads to the \( \mathcal{O}(\epsilon) \) equation in (17) becoming
\[
  \frac{d^2 u_1}{d\tau^2} + u_1 = \left( 2\omega_1 - \frac{3}{4} \right) \cos(\tau) - \frac{1}{4} \cos(3\tau), \quad u_1(0) = u_1'(0) = 0
\] (19)

NOTE that Equation (19) is solved using the same techniques for Equation (11), with the extra term \( 2\omega_1 \cos(\tau) \) coming from \( -2\omega_1 u_0'' \).

In order to eliminate the \( \cos(\tau) \) term on the right-hand side of (19) we can let \( \omega_1 = \frac{3}{8} \) which produces
\[
  \frac{d^2 u_1}{d\tau^2} + u_1 = -\frac{1}{4} \cos(3\tau)
\]

We can again use the Method of Undetermined Coefficients and the initial conditions to show that the solution to the above equation is
\[
  u_1(\tau) = \frac{1}{32}[\cos(3\tau) - \cos(\tau)] \text{ where } \tau = t + \frac{3}{8}\epsilon t + \ldots
\] (20)

A first-order, uniformly-valid perturbation solution of Duffing’s Equation (6) is \( u_0(\tau) + \epsilon u_1(\tau), \)
\[
  u(\tau) = \cos(\tau) + \frac{1}{32}\epsilon[\cos(3\tau) - \cos(\tau)] \text{ where } \tau = t + \frac{3}{8}\epsilon t + \ldots
\] (21)
A graph of (21) versus time for a typical value of $\epsilon = 0.1$ on the interval $0 \leq t \leq \frac{1}{\epsilon^2}$ is shown below. NOW what do you notice?

Here’s a graph of the difference between $u(\tau)$ and $u_0(\tau)$ which equals $\epsilon u_1(\tau)$ on the same interval $0 \leq t \leq \frac{1}{\epsilon^2}$

EXPLAIN the significance of the above Figures

**Homework Questions for Math 395: Applied Mathematics due TUE APR 7**

For each of the problems, use a Poincaré-Lindstedt method to obtain a 2-term perturbation approximation to the following problems. Also produce a graph (on the same axes) of $y_0(t)$ and/or $y_0(t) + \epsilon y_1(t)$ on the interval $0 \leq t \leq \frac{1}{\epsilon^2}$ for a reasonably small value of $\epsilon$ which indicates that your solution is uniformly valid for all $t$ values.

GROUP 1: Logan, page 101, Question 8(a)
(a) $y'' + y = \epsilon yy'^2$, \hspace{2mm} $y(0) = 1$, \hspace{2mm} $y'(0) = 0$

GROUP 2: Logan, page 101, Question 8(b)
(b) $y'' + 9y = 3\epsilon y^3$, \hspace{2mm} $y(0) = 0$, \hspace{2mm} $y'(0) = 1$

GROUP 3: Logan, page 101, Question 8(c)
(b) $y'' + y = \epsilon(1 - y'^2)$, \hspace{2mm} $y(0) = 1$, \hspace{2mm} $y'(0) = 0$