

Class 9: Tuesday March 24

TITLE How Regular Perturbations on ODEs Can Go Wrong
CURRENT READING Logan, Sections 2.1.2 and 2.1.3

SUMMARY
This week we continue looking at regular perturbations in differential equations and stumble upon what can go wrong. We'll be introduced to a method to still produce reasonable perturbation solutions called the Poincaré-Lindstedt method.

RECALL
Given the IVP which models an object falling through a medium with air resistance proportional to current velocity squared

$$m\frac{dv}{d\tau} = -av + bv^2, \qquad v(0) = V_0 \tag{1}$$

We can non-dimensionalize the model using the scalings

$$y = \frac{v}{V_0}, \qquad t = \frac{\tau}{m/a} \tag{2}$$

to produce

$$\frac{dy}{dt} = -y + \epsilon y^2, \qquad y(0) = 1 \text{ where } \epsilon = \frac{bV_0}{a} \ll 1 \tag{3}$$

Similarly, given the following model for a nonlinear spring-mass oscillator

$$m\frac{d^2y}{d\tau^2} = -ky - ay^3, \qquad y(0) = A, \quad \frac{dy}{d\tau}(0) = 0 \tag{4}$$

we can non-dimensionalize it using the scalings

$$u = \frac{y}{A}, \qquad t = \frac{\tau}{\sqrt{m/k}} \tag{5}$$

to produce

$$\frac{d^2u}{dt^2} = -u - \epsilon u^3, \qquad u(0) = 1, \quad u'(0) = 0 \text{ where } \epsilon = \frac{aA^2}{k} \ll 1 \tag{6}$$

The IVP in (6) is known as Duffing's Equation and has no known exact solution.

If we assume a perturbation series solution of the form

$$u(t) = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \ldots \tag{7}$$

then we will produce a series of differential equations (with initial conditions) of various orders in epsilon...

$\boxed{\text{EXAMPLE}}$

Let's show what the systems we get are:

The $\mathcal{O}(1)$ equation is

$$\frac{d^2u_0}{dt^2} + u_0 = 0, \quad u_0(0) = 1, \quad u_0'(0) = 0 \tag{8}$$

The $\mathcal{O}(\epsilon)$ equation is

$$\frac{d^2u_1}{dt^2} + u_1 = -u_0^3, \quad u_1(0) = 0, \quad u_1'(0) = 0 \tag{9}$$

The solution to the leading order IVP, the $\mathcal{O}(1)$ term in (7) is

$$u_0(t) = \cos(t) \tag{10}$$

which means that the $\mathcal{O}(\epsilon)$ equation becomes

$$\frac{d^2u_1}{dt^2} + u_1 = -\cos^3(t), \quad u_1(0) = 0, \quad u_1'(0) = 0$$

But using the common trigonometric identity $\cos(3t) = 4\cos^3(t) - 3\cos(t)$ the first-order equation (9) becomes

$$\frac{d^2u_1}{dt^2} + u_1 = -\frac{3}{4}\cos(t) - \frac{1}{4}\cos(3t), \quad u_1(0) = 0, \quad u_1'(0) = 0 \tag{11}$$

which can be solved using the Method of Undetermined Coefficients (assume a solution of the form $A\cos(t) + B\sin(t) + C\cos(3t) + Dt\cos(t) + Et\sin(t)$ which produces the following solution (after applying the initial conditions)

$$u_1(t) = \frac{1}{32}\cos(3t) - \frac{1}{32}\cos(t) - \frac{3}{8}t\sin(t) \tag{12}$$

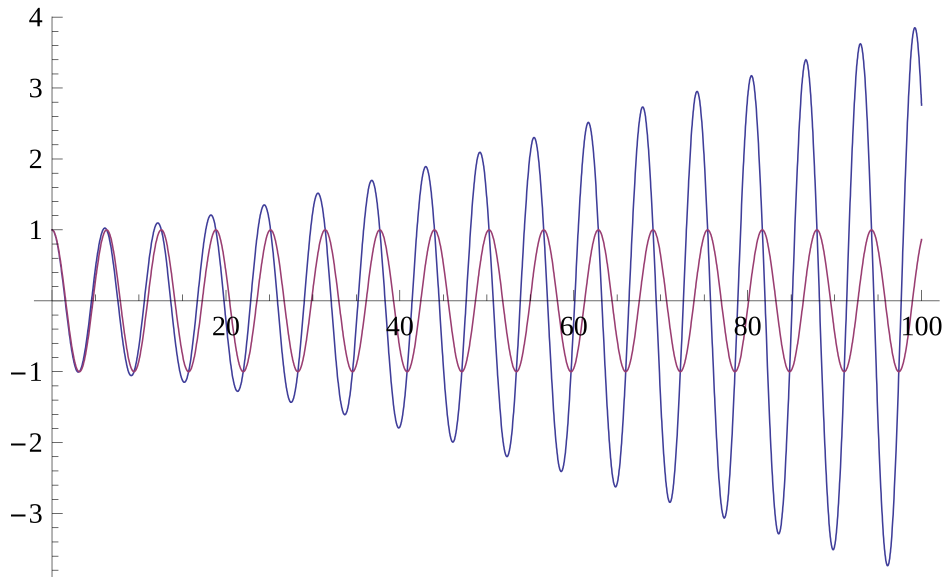
$\boxed{\text{EXAMPLE}}$

We can confirm this above result.

Exercise

Confirm that the given functions in (10) and (12) are indeed the solution to the IVPs in (8) and (9), respectively.

Consider a graph of $u_0(t)$ and $u_0(t) + \epsilon u_1(t)$ plotted versus time for a typical value of $\epsilon = 0.1$ on the interval $0 \leq t \leq \dfrac{1}{\epsilon^2}$. What do you notice?



Therefore $\epsilon u_1(t)$ is NOT much less than $u_0(t)$ for all time. Can you explain what happens as t gets larger and larger? Is it possible to estimate the value of t where "trouble" begins?

Explain the significance of the Figure

The Poincaré-Lindstedt Method

In this technique the perturbation series is chosen to be

$$u(\tau) = u_0(\tau) + \epsilon u_1(\tau) + \epsilon^2 u_2(\tau) + \dots \tag{13}$$

where $\tau = \omega t$ and

$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \dots \tag{14}$$

We can choose $\omega_0 = 1$ since it is the frequency of the solution given in (10) to the leading-order problem in Equation (9).

Using the new scalings given in (13) and (14) we can transform (6) into

$$\omega^2 \frac{d^2u}{d\tau^2} = -u - \epsilon u^3, \qquad u(0) = 1 \quad u'(0) = 0 \tag{15}$$

$\boxed{\text{EXAMPLE}}$

First let's show how we get from (6) to (15)

and then we can show that the equations we get are:

The $\mathcal{O}(1)$ equations are

$$\frac{d^2u_0}{d\tau^2} + u_0 = 0, \quad u_0(0) = 1, \quad u_0'(0) = 0 \tag{16}$$

The $\mathcal{O}(\epsilon)$ equations are

$$\frac{d^2u_1}{d\tau^2} + u_1 = -2\omega_1 u_0'' - u_0^3, \quad u_1(0) = 0, \quad u_1'(0) = 0 \tag{17}$$

The solution to (16), $\frac{d^2u_0}{d\tau^2} + u_0 = 0, \quad u_0(0) = 1, \quad u_0'(0) = 0$, is similar to the solution from (8) which turns out to be

$$u_0(\tau) = \cos(\tau) \tag{18}$$

which leads to the $\mathcal{O}(\epsilon)$ equation in (17) becoming

$$\frac{d^2u_1}{d\tau^2} + u_1 = \left(2\omega_1 - \frac{3}{4}\right)\cos(\tau) - \frac{1}{4}\cos(3\tau), \quad u_1(0) = u_1'(0) = 0 \tag{19}$$

NOTE that Equation (19) is solved using the same techniques for Equation (11), with the extra term $2\omega_1\cos(\tau)$ coming from $-2\omega_1 u_0''$.

In order to eliminate the $\cos(\tau)$ term on the right-hand side of (19) we can let $\omega_1 = \dfrac{3}{8}$ which produces

$$\frac{d^2u_1}{d\tau^2} + u_1 = -\frac{1}{4}\cos(3\tau)$$

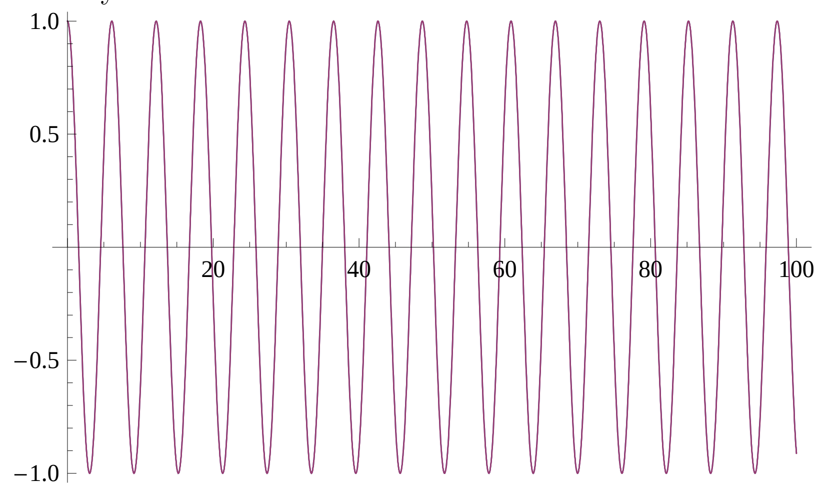
We can again use the Method of Undetermined Coefficients and the initial conditions to show that the solution to the above equation is

$$u_1(\tau) = \frac{1}{32}[\cos(3\tau) - \cos(\tau)] \text{ where } \tau = t + \frac{3}{8}\epsilon t + \ldots \tag{20}$$

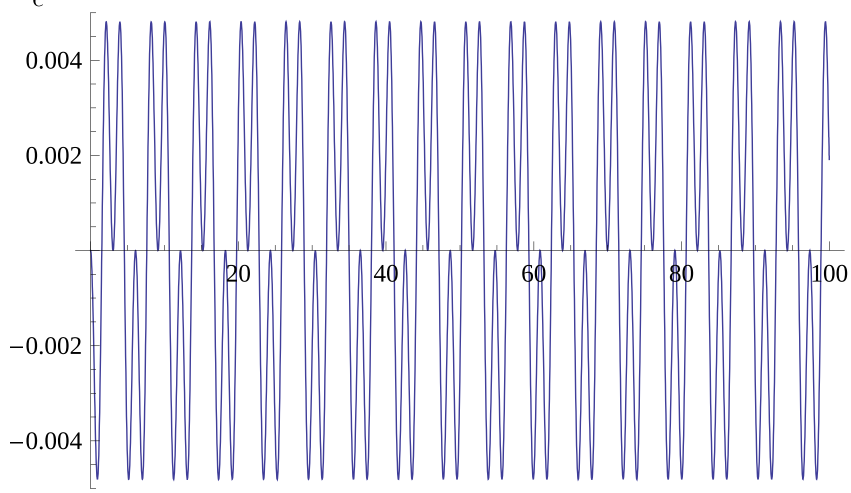
A first-order, uniformly-valid perturbation solution of Duffing's Equation (6) is $u_0(\tau)+\epsilon u_1(\tau)$,

$$u(\tau) = \cos(\tau) + \frac{1}{32}\epsilon[\cos(3\tau) - \cos(\tau)] \text{ where } \tau = t + \frac{3}{8}\epsilon t + \ldots \tag{21}$$

A graph of (21) versus time for a typical value of $\epsilon = 0.1$ on the interval $0 \le t \le \dfrac{1}{\epsilon^2}$ is shown below. NOW what do you notice?



Here's a graph of the difference between $u(\tau)$ and $u_0(\tau)$ which equals $\epsilon u_1(\tau)$ on the same interval $0 \le t \le \dfrac{1}{\epsilon^2}$



EXPLAIN the significance of the above Figures

Homework Questions for Math 395: Applied Mathematics due TUE APR 7

For each of the problems, use a Poincaré-Lindstedt method to obtain a 2-term perturbation approximation to the following problems. Also produce a graph (on the same axes) of $y_0(t)$ and/or $y_0(t) + \epsilon y_1(t)$ on the interval $0 \le t \le \dfrac{1}{\epsilon^2}$ for a reasonably small value of ϵ which indicates that your solution is uniformly valid for all t values.

GROUP 1: Logan, page 101, Question 8(a)
(a) $y'' + y = \epsilon y y'^2, \quad y(0) = 1, \quad y'(0) = 0$

GROUP 2: Logan, page 101, Question 8(b)
(b) $y'' + 9y = 3\epsilon y^3, \quad y(0) = 0, \quad y'(0) = 1$

GROUP 3: Logan, page 101, Question 8(c)
(b) $y'' + y = \epsilon(1 - y'^2), \quad y(0) = 1, \quad y'(0) = 0$