Applied Mathematics

Math 395 Spring 2009 | Fowler 301 Tue 3:00pm - 4:25pm

 | http://faculty.oxy.edu/ron/math/395/09/

Class 8: Tuesday March 17

TITLE Singular Perturbations of Ordinary Differential Equations
CURRENT READING Logan, Section 2.1.1 and 2.2.1

SUMMARY
Last week we looked at what happens when small perturbations are introduced into algebraic equations. This week let's begin looking at perturbations in differential equations.

The Perturbation Series

Consider the differential equation for a body of mass m moving in a straightline with initial velocity V_0 subject to a resistive force (*cough* bomb drop *cough*) results in the following IVP

$$m\frac{dv}{d\tau} = -av + bv^2, \qquad v(0) = V_0 \tag{1}$$

We can non-dimensionalize the IVP using

$$y = \frac{v}{V_0}, \qquad t = \frac{\tau}{m/a} \tag{2}$$

to produce

$$\frac{dy}{dt} = -y + \epsilon y^2, \qquad y(0) = 1 \text{ where } \epsilon = \frac{bV_0}{a} \ll 1 \tag{3}$$

The ODE in (3) is in a class of differntial equations called Bernoulli Equations (named after the famous Bernoulli Brothers Jacob and Johann who were instrumental in the development of Fluid Mechanics) where "the trick" is to change variables through the substitution $w = \frac{1}{y}$ to produce a linear equation which is easily solved to produce the exact solution

$$y(t) = \frac{e^{-t}}{1 + \epsilon(e^{-t} - 1)} \tag{4}$$

Exercise

Confirm that the given solution in (4) is indeed the solution to the IVP in (3)

EXAMPLE

Let's also show that we can do a Taylor Expansion of the exact solution in (4) to produce an approximation which looks like

$$y_{exact}(t) = e^{-t} + \epsilon(e^{-t} - e^{-2t}) + \epsilon^2(e^{-t} - 2e^{-2t} + e^{-3t}) + \ldots \tag{5}$$

Perturbation Series Solution

Let's assume that the solution to the IVP in (3), $y' = -y + \epsilon y^2, \quad y(0) = 1$ has a perturbation series solution like we have been assuming previoulsy for algebraic equations.

$$y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \ldots \tag{6}$$

Then we will produce a series of differential equations (with initial conditions) of various orders in epsilon...

We can show that the systems we get are:

The $\mathcal{O}(1)$ equation is

$$y_0' = -y_0, \quad y(0) = 1 \tag{7}$$

The $\mathcal{O}(\epsilon)$ equation is

$$y_1' = -y_1 + y_0^2, \quad y_1(0) = 0 \tag{8}$$

The $\mathcal{O}(\epsilon^2)$ equation is

$$y_2' = -y_2 + 2y_0 y_1, \quad y_2(0) = 0 \tag{9}$$

Let's solve these individual IVPs for $y_0(t)$, $y_1(t)$ and $y_2(t)$ and show that our 3-term approximation $y_{approx}(t)$ to the solution of (3), $y' = -y + \epsilon y^2, \quad y(0) = 1$

$$y_{approx}(t) = e^{-t} + \epsilon(e^{-t} - e^{-2t}) + \epsilon^2(e^{-t} - 2e^{-2t} + e^{-3t}) + \ldots \tag{10}$$

Look familiar?

The point here is that $y_{exact} - y_{approx} = m_1(t)\epsilon^3 + m_2\epsilon^4 + \ldots$ for $t > 0$ where m_1, m_2, ... are bounded functions so that as $\epsilon \to 0$ this difference will go to zero for all positive values of t.

Homework Questions for Math 395: Applied Mathematics due TUE MAR 17

GROUP 1: Logan, page 100, Question 2
Consider the initial value problem

$$u'' - u = \epsilon t u, \qquad t > 0, \quad u(0) = 1, \quad u'(0) = -1$$

Find a 2-term perturbation approximation for $0 \le \epsilon \ll 1$ and compare graphically to a 6-term Taylor series approximation (centered at $t = 0$) where $\epsilon = .04$. Use a numerical DE solver to find an accurate numerical solution and compare.

GROUP 2: Logan, page 103, Question 14
Consider the problem

$$\frac{dy}{dt} = e^{-\epsilon/y}, \quad y(0) = 1$$

Find the leading order approximation and derive initial value problems for the $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$ terms. DO NOT SOLVE! Write down the exact solution of the problem.

GROUP 3: Logan, page 103, Question 16
Consider the initial value problem $y'' = \epsilon t y, \quad 0 \le \epsilon \ll 1, \quad y(0) = 0, \quad y'(0) = 1$. Using regular perturbation theory obtain a 3-term approximate solution on $t > 0$. Does the approximation satisfy the differential equation uniformly (i.e. for all values) $t \ge 0$ as $\epsilon \to 0^+$?