
Applied Mathematics

Math 395 Spring 2009
©2009 Ron Buckmire

Fowler 301 Tue 3:00pm - 4:25pm
<http://faculty.oxy.edu/ron/math/395/09/>

Class 7: Tuesday March 1

TITLE Perturbations on Differential Equations

CURRENT READING Logan, Section 2.1.1 and 2.2.1

SUMMARY

Last week we looked at what happens when small perturbations are introduced into algebraic equations. This week let's begin looking at perturbations in differential equations.

The Perturbation Series

Consider the differential equation for a body of mass m moving in a straightline with initial velocity V_0 subject to a resistive force (*cough* bomb drop *cough*) results in the following IVP

$$m \frac{dv}{d\tau} = -av + bv^2, \quad v(0) = V_0 \quad (1)$$

We can non-dimensionalize the IVP using

$$y = \frac{v}{V_0}, \quad t = \frac{\tau}{m/a} \quad (2)$$

to produce

$$\frac{dy}{dt} = -y + \epsilon y^2, \quad y(0) = 1 \text{ where } \epsilon = \frac{bV_0}{a} \ll 1 \quad (3)$$

The ODE in (3) is in a class of differential equations called Bernoulli Equations (named after the famous Bernoulli Brothers Jacob and Johann who were instrumental in the development of Fluid Mechanics) where “the trick” is to change variables through the substitution $w = \frac{1}{y}$ to produce a linear equation which is easily solved to produce the exact solution

$$y(t) = \frac{e^{-t}}{1 + \epsilon(e^{-t} - 1)} \quad (4)$$

Exercise

Confirm that the given solution in (4) is indeed the solution to the IVP in (3)

EXAMPLE

Let's also show that we can do a Taylor Expansion of the exact solution in (4) to produce an approximation which looks like

$$y_{exact}(t) = e^{-t} + \epsilon(e^{-t} - e^{-2t}) + \epsilon^2(e^{-t} - 2e^{-2t} + e^{-3t}) + \dots \quad (5)$$

Perturbation Series Solution

Let's assume that the solution to the IVP in (3), $y' = -y + \epsilon y^2$, $y(0) = 1$ has a perturbation series solution like we have been assuming previously for algebraic equations.

$$y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \dots \quad (6)$$

Then we will produce a series of differential equations (with initial conditions) of various orders in epsilon...

We can show that the systems we get are:

The $\mathcal{O}(1)$ equation is

$$y_0' = -y_0, \quad y_0(0) = 1 \quad (7)$$

The $\mathcal{O}(\epsilon)$ equation is

$$y_1' = -y_1 + y_0^2, \quad y_1(0) = 0 \quad (8)$$

The $\mathcal{O}(\epsilon^2)$ equation is

$$y_2' = -y_2 + 2y_0 y_1, \quad y_2(0) = 0 \quad (9)$$

Let's solve these individual IVPs for $y_0(t)$, $y_1(t)$ and $y_2(t)$ and show that our 3-term approximation $y_{approx}(t)$ to the solution of (3), $y' = -y + \epsilon y^2$, $y(0) = 1$

$$y_{approx}(t) = e^{-t} + \epsilon(e^{-t} - e^{-2t}) + \epsilon^2(e^{-t} - 2e^{-2t} + e^{-3t}) + \dots \quad (10)$$

Look familiar?

The point here is that $y_{exact} - y_{approx} = m_1(t)\epsilon^3 + m_2\epsilon^4 + \dots$ for $t > 0$ where m_1, m_2, \dots are bounded functions so that as $\epsilon \rightarrow 0$ this difference will go to zero for all positive values of t .

Homework Questions for Math 395: Applied Mathematics due TUE MAR 17
--

GROUP 1: Logan, page 111, 1(b)

Find a 2-term approximation to the roots of $\epsilon x^3 + x - 2 = 0$.

GROUP 2: Logan, page 101, page 6

Find a 3-term approximation to the roots of $\exp(\epsilon x) = x^2 - 1$ of the form $x = x_0 + \epsilon x_1 + \epsilon^2 x_2$.

GROUP 3: Logan, page 101, page 7

Find a 3-term approximation to the real solution of three roots of $(x + 1)^3 = \epsilon x$.