Applied Mathematics

Math 395 Spring 2009 | Fowler 301 Tue 3:00pm - 4:25pm

 | http://faculty.oxy.edu/ron/math/395/09/

Class 6: Tuesday February 24

TITLE Introduction to Perturbations on Algebraic Equations
CURRENT READING Logan, Section 2.1.1 and 2.2.1

SUMMARY
This week we will be introduced into the wonderful world of perturbations. We shall begin by looking at what can happen to simple algebraic equations when a small term is included.

The Perturbation Series

Consider the differential equation

$$F(t, y, y', y'', \epsilon) = 0 \text{ where } t \in I \text{ and } \epsilon \ll 1 \tag{1}$$

We approximate the solution to (1) by a perturbation series given by

$$y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t) + \dots \tag{2}$$

where the functions y_0, y_1 and y_2, ... are found by substituting the perturbation series in (2) into (1) to produce equations that correspond to the leading order, first order and second order terms (i.e. $\mathcal{O}(1)$, $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$ respectively).

The leading order term $y_0(t)$ in the perturbation series is called the solution to the unperturbed version of (2) when $\epsilon = 0$, in other words

$$F(t, y_0, y_0', y_0'', 0) = 0 \text{ where } t \in I \tag{3}$$

The basic idea is that only a couple terms of the perturbation series need to be computed and that the difference between the exact solution to (2) and the perturbation series goes to zero uniformly on the interval I (in other words for every value of $t \in I$. The higher-order terms are seen as correction terms which are increasingly smaller for some relatively small value of ϵ.

NOTE
The leading order problem (3) must be solvable and the terms in the perturbation series given in (2) must maintain their relative sizes on the given interval for the problem to be considered a **regular perturbation** problem. Additionally, the qualitative nature of the unperturbed solution should resemble the perturbed solution. (In other words there should not be a radical change if $\epsilon \to 0$) Otherwise, it is known as a **singular perturbation**.

First we shall look at regular perturbations in algebraic equations...

Regular Perturbation Of Algebraic Equations

Consider the equation

$$x^2 + 2\epsilon x - 3 = 0 \tag{4}$$

Considering a perturbation series of the form $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots$ and writing down equations of each order, produces

The $\mathcal{O}(1)$ equation is

$$x_0^2 - 3 = 0 \tag{5}$$

The $\mathcal{O}(\epsilon)$ equation is

$$2x_0(x_1 + 1) = 0 \tag{6}$$

The $\mathcal{O}(\epsilon^2)$ equation is

$$x_1^2 + 2x_0x_2 + 2x_1 = 0 \tag{7}$$

Solving (5), (6) and (7) in order produces the solutions $x_0 = \pm\sqrt{3}$, $x_1 = -1$ and $x_2 = \pm\dfrac{1}{2\sqrt{3}}$ that corresponds to

$$x = \sqrt{3} - \epsilon + \frac{1}{2\sqrt{3}}\epsilon^2 + \dots \text{ and } x = -\sqrt{3} - \epsilon - \frac{1}{2\sqrt{3}}\epsilon^2 + \dots$$

EXAMPLE

Let's show this result

NOTE

This result can also be shown by using the quadratic formula on the equation in (4) and using a Taylor Expansion...

Singular Perturbation Of Algebraic Equations

In the case where the unperturbed ($\epsilon = 0$) problem has a quantitatively different nature than the perturbed problem we are usually dealing with a singular perturbation problem.

EXAMPLE

Consider Example 2.6 on page 104 of Logan

$$\epsilon x^2 + 2x + 1 = 0, \qquad 0 \le \epsilon \ll 1 \tag{8}$$

Notice what happens when you consider the unperturbed version of (8) by solving the problem with $\epsilon = 0$.

The $\mathcal{O}(\epsilon^0)$ equation is

$$2x_0 + 1 = 0 \tag{9}$$

so that $x_0 = -\dfrac{1}{2}$ and the leading order equation is **linear** instead of quadratic! It will produce fewer solutions than the original perturbed problem in (8). That is a tell-tale sign of a regular perturbation problem.

Note that we can still obtain the higher-order equations to obtain values for x_1 and x_2...

The $\mathcal{O}(\epsilon)$ equation is

$$x_0^2 + 2x - 1 = 0 \tag{10}$$

The $\mathcal{O}(\epsilon^2)$ equation is

$$2x_1x_0 + 2x_2 = 0 \tag{11}$$

which leads to only one solution to the given perturbed quadratic equation in (8)

$$x = -\frac{1}{2} - \frac{1}{8}\epsilon - \frac{1}{16}\epsilon^2 + \ldots \tag{12}$$

To find the other solution we must "re-balance" terms by making a new estimate (13) for the perturbation series and plug it into (8) instead

$$x = x_0 + \epsilon^\nu x_1 + \epsilon^{2\nu} x_2 + \ldots \tag{13}$$

In general we can really just keep the first two terms in (13) and show that this gives us the same $\mathcal{O}(1)$ equation as given before in (9) but the following equation for the next order term produces

$$x_0^2\epsilon + x_1^2\epsilon^{2\nu+1} + 2x_0x_1\epsilon^{\nu+1} + 2x_1\epsilon^\nu = 0 \tag{14}$$

The Hard Way: Dominant Balancing

Write a number under each of the four terms in equation (14). What follows now is that we have to look at all the posibilities of balancing any two of these terms with each other and solving for ν. The correct balancing will produce a well-ordered perturbation series. There are $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ possibilities (i.e. six).

$$x_0^2\epsilon + x_1^2\epsilon^{2\nu+1} + 2x_0x_1\epsilon^{\nu+1} + 2x_1\epsilon^{\nu} = 0$$

The correct balancing is TERM 2 and TERM 4 which leads to $\nu = -1$. This results in the following equations

$$\begin{aligned} 2x_0 + 1 + 2x_0x_1 &= 0 \qquad \mathcal{O}(1) \\ 2x_1 + x_1^2 &= 0 \qquad \mathcal{O}\left(\frac{1}{\epsilon}\right) \end{aligned}$$

which produces the second root to equation (8), i.e. $\epsilon x^2 + 2x + 1 = 0$ to be

$$x = -\frac{2}{\epsilon} + \frac{1}{2} + \dots \tag{15}$$

The Easier Way: Revert To A Regular Perturbation Problem
We could have just looked at the three terms in $\epsilon x^2 + 2x + 1 = 0$ and looked at the two possible balancings left in which you keep two of the terms in the equation and ignore the third because it is very much smaller than the other two when $\epsilon \ll 1$.

The first balancing is ϵx^2 with 1 which produces $x \approx \mathcal{O}(1/\sqrt{\epsilon})$ which means that the first and third terms will be the same size while the middle term will be much larger so therefore it can NOT be ignored. Thus this balncing is rejected.

The second balancing is of ϵx^2 with 1 and finds that $x = \mathcal{O}\left(\frac{1}{\epsilon}\right)$. This is the only one that "makes sense." In other words, the first and second terms will be $\mathcal{O}(1/\epsilon)$ and the third term will be $\mathcal{O}(1)$ which is much *smaller* when ϵ is very small so this term can be ignored. This means that one could re-scale x so that $y = \dfrac{x}{1/\epsilon}$ and re-substitute into (8) to produce

$$y^2 + 2y + \epsilon = 0 \tag{16}$$

which is a regular perturbation problem that can be solved using a perturbation series of the form $y = y_0 + \epsilon y_1 + y_2\epsilon^2 + \ldots$.

Exercise
Use this sclaing (remembering that $y = \epsilon x$) and show that one obtains the same result as given in (8) in the space below. In fact, go further and obtain a 3-term expansion for this root.

The Easiest Way: Quadratic Formula Plus Taylor Expansion
Find a three-term expansion for the roots of $\epsilon x^2 + 2x + 1 = 0$ by using the quadratic formula and the Taylor Expansion for $\sqrt{1+\epsilon} \approx 1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \mathcal{O}(\epsilon^3)$