Applied Mathematics

Math 395 Spring 2009 — Fowler 301 Tue 3:00pm - 4:25pm

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Class 3: **Tuesday February 3**

TITLE Introduction to Scaling
CURRENT READING Logan, Section 1.2

SUMMARY
This week we will be introduced to the concept of characteristic scales and the importance of scaling in real-world problems.

DEFINITION: **Scaling**

The process of selecting new, usually dimensionless variables and re-formulating the problem in terms of those new variables (Logan 19). Sometimes the process is called **nondimensionalization**. The Buckingham pi Theorem assures us that we can always find a non-dimensional (scaled) version of a given problem.

Most commonly, scaling is used on the time variable. Many real world processes occur over various time scales, such as in a chemical reaction where one might have small changes in concentration over a relatively long period of time, and the suddenly a tipping point is reached and a very fast change in concentration happens very very quickly.

Generally one does this by selecting a characteristic time value t_c and making a dimensionless version of time $\bar{t}$ by using the following equation $\bar{t} = \dfrac{t}{t_c}$

Or in biology one can have different lengths which vary incredibly widely as one considers "genes, proteins, cells, organs, organisms, communities and ecocsystems" (Logan 20).

Population Models

The Malthus Population model states that the growth rate of a population is proportional to its current population, i.e.

$$\frac{dP}{dt} \propto P \Rightarrow \frac{dP}{dt} = kP$$

This results in exponential growth! It also means that the per capita growth rate (the rate divided by the total population) is a constant value k.

The Verhulst Population model (sometimes known as the Logistic Model) is a modification of the Malthusian model which changes the per capita growth rate from a constant to a rate that decreases linearly as population increases to a maximum value, called the carrying capacity M, or

$$\frac{1}{P}\frac{dP}{dt} = k\left(1 - \frac{P}{M}\right)$$

EXAMPLE

Consider

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right), \qquad P(0) = P_0 \tag{1}$$

Let's scale this problem by producing a non-dimensional version of the problem We need to select dimensionless versions of the dependent (P) and independent variables (t). We'll form a characteristic value P_c and t_c from the given constants in the problem: P_0, M and k.

Use $P_c = M$ and $t_c = \frac{1}{k}$ so that $\bar{P} = \dfrac{P}{m}$ and $\bar{t} = \dfrac{t}{\frac{1}{k}}$

This produces the nondimensional version of (1) which looks like

$$\frac{d\bar{P}}{d\bar{t}} = \bar{P}(1 - \bar{P}), \qquad \bar{P}(0) = \alpha \tag{2}$$

where $\alpha = \dfrac{P_0}{K}$ which is a dimensionless constant. Note that the solution to (2) is an unknown function $\bar{P}(\bar{t})$ while the solution to the original problem (1) is an unknown function $P(t)$. The relationship between the two is $P(t) = P_c\bar{P}(\bar{t})$ and $t = t_c\bar{t}$.

Exercise

Show that if you select a different scaling of $P_c = P_0$ and $t_c = \frac{1}{k}$ one obtains the following dimensionless equation:

$$\frac{d\bar{P}}{d\bar{t}} = \bar{P}(1 - \beta\bar{P}), \qquad \bar{P}(0) = 1$$

where $\beta = \dfrac{P_0}{M}$ is a dimensionless constant.