Applied Mathematics

Math 395 Spring 2009 — Fowler 301 Tue 3:00pm - 4:25pm

 http://faculty.oxy.edu/ron/math/395/

Class 2: **Tuesday January 27**

TITLE Dimenionsl Analysis, Continued
CURRENT READING Logan, Section 1.1.1

SUMMARY
This week we will do some more complicated dimensional analysis and look at the Buckingham Pi Theorem more formally.

RECALL
The informal statement of the Pi Theorem: "Given a physical law that gives a relation among a certain number of *dimensioned* quantities, then there is an equivalent law that can be expressed as a relation among certain **dimensionless** quantities" (Logan 5).

The Buckingham Pi Theorem

On page 10 of Logan the Pi Theorem is formally presented:

> Let
> $$f(q_1, q_2, q_3, \dots, q_m) = 0 \tag{1}$$
> be a unit-free physical law that relates the dimensional quantities $q_1, q_2, \dots, q_m$. Let $L_1, L_2, \dots, L_n$ (with $n < m$) be fundamental dimensions with
> $$[q_i] = L_1^{a_{1i}} L_2^{a_{2i}} \dots L_n^{a_{ni}}, \text{ where } i = 1, \dots, m$$
> and let $r = \text{rank}(A)$, where A is an $n \times m$ dimension matrix with entries a_{ij}. THEN there exists $m-r$ independent dimensionless quantities $\pi_1, \pi_2, \pi_3, \dots, \pi_{m-r}$ that can be formed from $q_1, q_2, q_3, \dots q_m$ and the physical law (1) is equivalent to another equation
> $$F(\pi_1, \pi_2, \pi_3, \dots, \pi_{m-r}) = 0 \tag{2}$$
> expressed only in terms of **dimensionless** variables.

What does this all mean? Let's look at the theroem closely:

There are really two parts to it

(i) Among the quantities $q_1, q_2, \dots, q_m$ there are $m-r$ independent dimensionless variables that can be formed, where r is the rank of the dimension matrix A

(ii) If π_1, π_2, ..., π_{m-r} are the $m-r$ dimensionless variables, then $f(q_1, q_2, q_3, \dots, q_m) = 0$ (1) is equivalent to a physical law of the form $F(\pi_1, \pi_2, \pi_3, \dots, \pi_{m-r}) = 0$ (2).

Let's look at the proof given on Page 13 of Logan and then explore Example 1.6.

Proving Statement (i) is straightforward. Once you set

$$\pi = q_1^{\alpha_1} q_2^{\alpha_2} q_2^{\alpha_3} \dots q_m^{\alpha_m} \tag{3}$$

We can obtain a homogeneous set of n linear equations in m variables α_1, α_2, ..., α_m. We are looking for the set of solutions that span the nullspace of an $n \times m$ matrix with rank r, which is $m - r$. (Note that Logan describes his matrices using n times m instead of the customary m rows and n columns some of you may be used to from Linear Algebra).

Proving Statement (ii) involves looking at Example 1.6 which is about the law $f(x, t, g) = x - \frac{1}{2}gt^2 = 0$ and showing that one can produce a new law $F(\pi_1) = 0$ which is non-dimensionless and equivalent. Let's do that.

$\boxed{\textbf{Exercise}}$

Example 1.5 on page 11. Consider a physical law of the form $f(t, r, u, e, k, c) = 0$ where t is time, e is energy, r is radial distance from the heat source, c is the heat capacity and k is thermal difusivity. Using standard dimensions of T, L, Θ and E and this specific order of varaiables, show that the dimension matrix looks like

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

and dimensional analysis to show that

$$\frac{e}{c}(kt)^{-3/2} g\left(\frac{r}{\sqrt{kt}}\right) = u$$

is a version of a physical law relating the quantities.