Applied Mathematics

Math 395 Spring 2009 — Fowler 301 Tue 3:00pm - 4:25pm

 — http://faculty.oxy.edu/ron/math/395/09/

Class 10: **Tuesday April 14**

TITLE Itroduction To Boundary Layers
CURRENT READING Logan, Sections 2.2.2 and 2.2.3

SUMMARY
This week we will be introduced to the concept of boundary layers, i.e. a solution to a boundary value problem which is only valid during certain regions of the independent variable.

CONSIDER
Given the boundary value problem

$$\epsilon \frac{d^2y}{dx^2} + (1+\epsilon)\frac{dy}{dx} + y = 0, \text{ where } \epsilon \ll 1 \text{ and } \quad 0 < x < 1 \text{ with } y(0) = 0, \quad y(1) = 1. \quad (1)$$

We'll asssume the usual regular perturbation solution of the form

$$y(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \ldots \quad (2)$$

then we will produce a series of differential equations (with BOUNDARY conditions) of various orders in epsilon which look like...

The $\mathcal{O}(1)$ equation is

$$\frac{dy_0}{dx} + y_0 = 0, \quad y_0(0) = 0, \quad y_0(1) = 1 \quad (3)$$

The $\mathcal{O}(\epsilon)$ equation is

$$\frac{dy_1}{dx} + y_1 = -y_0'' - y_0', \quad y_1(0) = 0, \quad y_1(1) = 0 \quad (4)$$

EXAMPLE

Let's solve these boundary value problems and see what happens. $y_0(x) = Ae^{-x}$ is the homogeneous solution of (3). What happens when you solve for A?

So, this means that this form will not work and we're looking at a **singular** perturbation problem and have to figure out something else.

Let's look again at the exact problem given in Equation (1)

$$\epsilon y'' + (1+\epsilon)y' + y = 0 \tag{5}$$

Notice that it has the form $ay''+by'+cy=0$ where a, b and c are constant coefficients which depend ϵ. Assuming the ansatz of $y = e^{rx}$ you should be able show that the r satisfies the following equation

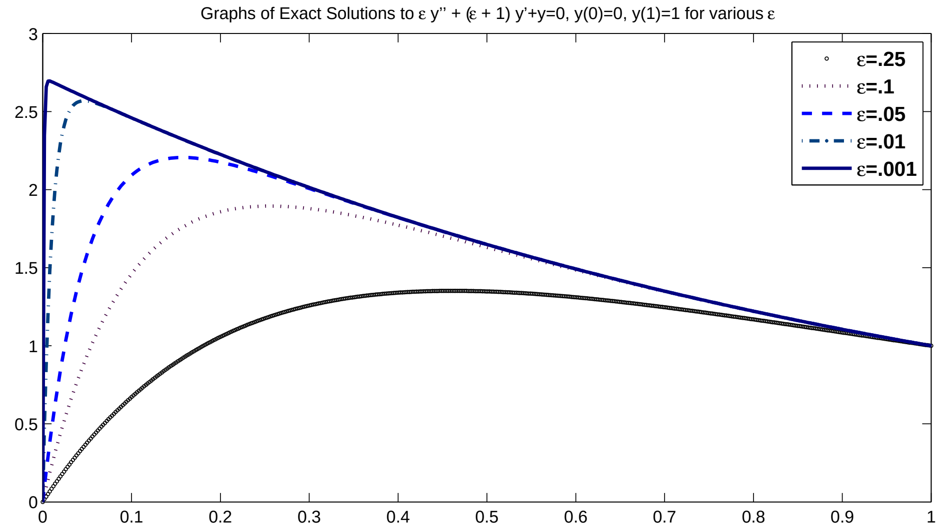
$$r = \frac{-(1+\epsilon) \pm (1-\epsilon)}{2\epsilon} \tag{6}$$

Exercise

Confirm that the exact solution to (5) by obtaining the values of r.

This means that the general exact solution to (5) is $y(x) = Ae^{-x} + Be^{-x/\epsilon}$ which when evaluated using the boundary conditions produces the exact solution to (1)

$$y(x) = \frac{1}{e^{-1} - e^{-1/\epsilon}}(e^{-x} - e^{-x/\epsilon} \tag{7}$$



Boundary Layer

Near one of the ends of interval of interest $0 \leq x \leq 1$ the exact solution $y(x)$ given in Equation (7) changes very rapidly, in a very thin which happens to be of size ϵ. This are of rapid change is called a **boundary layer**. Many real-world physical problems like fluid flow possess solutions which exhibit boundary layers.

We obtain the solution of the problem with a boundary layer by splitting the problem into two: an inner approximate solution $y_{inner}(x)$ and an outer approximate solution $y_{outer}(x)$. Each piece of the problem has a region of validity.

The Outer Problem

The outer solution is easier to find because it is valid when ϵ is ignored, i.e. in the range where $\mathcal{O}(\epsilon) < x \leq 1$, so you can let $\epsilon = 0$ in the original problem given in (1)

$$y'_{outer} + y_{outer} = 0, \quad y_{outer}(1) = 1 \tag{8}$$

Notice the inner boundary condition near $x = 0$ is ignored.

Solving the IVP in (8) gives us the solution $y_{outer}(x) = e^{1-x}$.

Exercise

You should check that the outer solution $y_{outer}(x) = e^{1-x}$ satisfies the IVP given in Equation (8)

The Inner Problem
The inner solution is valid when $0 \leq x < \mathcal{O}(\epsilon)$ which means its solves the problem

$$\epsilon y''_{inner} + (1+\epsilon)y'_{inner} + y_{inner} = 0, \quad y_{inner}(0) = 0 \tag{9}$$

However, we don't know how to solve this problem exactly since it is a second-order differential equation with only one condition on $y(x)$. We'll explore this further later.

We do, however know that the inner solution will have the form $y(x) = A(e^{-x} - e^{-x/\epsilon})$ from when we used our ansatz and found the values of r.

It turns out that $A = e^1$ so that the inner solution has the form $y(x) = e^{1-x} - e^{1-x/\epsilon}$. This function solves the ODE and the inner boundary condition.

However, since this function is only valid for small values of $x \ll 1$ this implies that $e^{1-x} \approx e$ so it can really be approximated by $y_{inner}(x) = e - e^{1-x/\epsilon}$

Exercise
You should check that the function $y_{inner}(x) = e - e^{1-x/\epsilon}$ satisfies the IVP given in Equation (9)