Class 10: Tuesday April 14

TITLE Introduction To Boundary Layers
CURRENT READING Logan, Sections 2.2.2 and 2.2.3

SUMMARY
This week we will be introduced to the concept of boundary layers, i.e. a solution to a boundary value problem which is only valid during certain regions of the independent variable.

CONSIDER
Given the boundary value problem

$$\epsilon \frac{d^2 y}{dx^2} + (1 + \epsilon) \frac{dy}{dx} + y = 0, \text{ where } \epsilon \ll 1 \text{ and } 0 < x < 1 \text{ with } y(0) = 0, \quad y(1) = 1. \quad (1)$$

We’ll assume the usual regular perturbation solution of the form

$$y(x) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \ldots \quad (2)$$

then we will produce a series of differential equations (with BOUNDARY conditions) of various orders in epsilon which look like...

The $O(1)$ equation is

$$\frac{dy_0}{dx} + y_0 = 0, \quad y_0(0) = 0, \quad y_0(1) = 1 \quad (3)$$

The $O(\epsilon)$ equation is

$$\frac{dy_1}{dx} + y_1 = -y_0'' - y_0', \quad y_1(0) = 0, \quad y_1(1) = 0 \quad (4)$$

EXAMPLE
Let’s solve these boundary value problems and see what happens. $y_0(x) = Ae^{-x}$ is the homogeneous solution of (3). What happens when you solve for $A$?

So, this means that this form will not work and we’re looking at a singular perturbation problem and have to figure out something else.
Let’s look again at the exact problem given in Equation (1)

\[ \epsilon y'' + (1 + \epsilon)y' + y = 0 \] (5)

Notice that it has the form \( ay'' + by' + cy = 0 \) where \( a, b \) and \( c \) are constant coefficients which depend \( \epsilon \). Assuming the ansatz of \( y = e^{rx} \) you should be able show that the \( r \) satisfies the following equation

\[ r = \frac{-(1 + \epsilon) \pm (1 - \epsilon)}{2\epsilon} \] (6)

**Exercise**

Confirm that the exact solution to (5) by obtaining the values of \( r \).

This means that the general exact solution to (5) is \( y(x) = Ae^{-x} + Be^{-x/\epsilon} \) which when evaluated using the boundary conditions produces the exact solution to (1)

\[ y(x) = \frac{1}{e^{-1} - e^{-1/\epsilon}}(e^{-x} - e^{-x/\epsilon}) \] (7)

**Boundary Layer**

Near one of the ends of interval of interest \( 0 \leq x \leq 1 \) the exact solution \( y(x) \) given in Equation (7) changes very rapidly, in a very thin which happens to be of size \( \epsilon \). This are of rapid change is called a boundary layer. Many real-world physical problems like fluid flow possess solutions which exhibit boundary layers.

We obtain the solution of the problem with a boundary layer by splitting the problem into two: an inner approximate solution \( y_{inner}(x) \) and an outer approximate solution \( y_{outer}(x) \). Each piece of the problem has a region of validity.
**The Outer Problem**

The outer solution is easier to find because it is valid when $\epsilon$ is ignored, i.e. in the range where $O(\epsilon) < x \leq 1$, so you can let $\epsilon = 0$ in the original problem given in (1)

$$y_{outer} + y_{outer} = 0, \quad y_{outer}(1) = 1 \quad (8)$$

Notice the inner boundary condition near $x = 0$ is ignored.

Solving the IVP in (8) gives us the solution $y_{outer}(x) = e^{1-x}$.

**Exercise**

You should check that the outer solution $y_{outer}(x) = e^{1-x}$ satisfies the IVP given in Equation (8)
The Inner Problem
The inner solution is valid when $0 \leq x < \mathcal{O}(\epsilon)$ which means its solves the problem

$$
\epsilon y''_{\text{inner}} + (1 + \epsilon)y'_{\text{inner}} + y_{\text{inner}} = 0, \quad y_{\text{inner}}(0) = 0 \quad (9)
$$

However, we don’t know how to solve this problem exactly since it is a second-order differential equation with only one condition on $y(x)$. We’ll explore this further later.

We do, however know that the inner solution will have the form $y(x) = A(e^{-x} - e^{-x/\epsilon})$ from when we used our ansatz and found the values of $r$.

It turns out that $A = e^1$ so that the inner solution has the form $y(x) = e^{1-x} - e^{1-x/\epsilon}$. This function solves the ODE and the inner boundary condition.

However, since this function is only valid for small values of $x \ll 1$ this implies that $e^{1-x} \approx e$ so it can really be approximated by $y_{\text{inner}}(x) = e - e^{1-x/\epsilon}$

**Exercise**
You should check that the function $y_{\text{inner}}(x) = e - e^{1-x/\epsilon}$ satisfies the IVP given in Equation (9)