Numerical Analysis

Math 370 Fall 1998 MWF 11:30am - 12:25pm
 Fowler 127

Class 17: **Monday October 14**

SUMMARY Analyzing Iterative Methods
READING Burden & Faires, 78–86

Asymptotic Rates of Convergence of Iterative Methods
If we have a sequence of approximations $\{p_n\}$ which converges to p and there exist positive constants α and λ so that

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda$$

then the sequence is said to **converge to p with order** α, with an **asymptotic error constant** λ. The method is said to be an "α order" method. **How is this definition different from "big oh" and "little oh"?**

We have considered ____________ iterative methods in this class so far...
Write them down below with their corresponding iterative step $p_n = g(p_{n-1})$

Let's rank these in order of how fast they converge to the root, i.e. in asymptotic order.

Question
How can we prove this order? Can we determine the order of these methods analytically? can we do it experimentally? How???

Summary
In other words the iterative method is said to be of order α if one can show a relationship like $|e_{n+1}| \approx \lambda |e_n|^{\alpha}$

Example

Consider the two sequences p_n and q_n which both to converge to zero.
However, p_n converges **linearly**, while q_n converges **quadratically**.

Let's assume that the two sequences have the same asymptotic error constant of $\lambda = 0.5$ and $p_0 = 1$. Use your calculator to generate the first 6 or so elements of each sequence and record these in the table below.

You should be able to come up with a formula which relates the n-th iterate to the initial guess (zeroth iterate) for both p_n and q_n and makes filling out the table easier....

Iterate n	Linear Convergence $p_n =$	Quadratic Convergence $q_n =$

Do you think it is a "big deal" whether a method is linearly convergent versus quadratically convergent?

Bisection Method

Derive the error formula for the bisection method and write it below. In other words, get an expression for e_{n+1} in terms of e_n and or n.

Therefore, Bisection is a ____________ method, with $\lambda =$ ______ and $\alpha =$ ______

Fixed Point Iteration

We shall derive the asymptotic rate of convergence for Functional Iteration.

$p_{n+1} = g(p_n)$ and $e_{n+1} = p - p_{n+1}$ and $e_n = p - p_n$ therefore

$p_{n+1} = g(p - e_n) =$

Picard Iteration is a ____________ method with $\alpha =$ __________________

Newton's Method

In a similar fashion, we shall derive the asymptotic rate of convergence for Newton's Method and fill-in the table below

Method	Order of Convergence	Error Formula
Bisection		
False Position	1.442	
Secant	$\frac{1+\sqrt{5}}{2} \approx 1.618$	
Newton's		
Picard		

NOTE: The above only apply for simple roots (i.e. a root of multiplicity 1).

Definition

A root r of an equation $f(r) = 0$ has multiplicity m if and only if $0 = f(r) = f'(r) = \cdots = f^{(m-1)}(r) = 0$ but $f^{(m)}(r) \neq 0$.

For roots of multiplicity $m > 1$ Newton's Method has the relationship that $|e_{n+1}| \approx \dfrac{M-1}{M}|e_n|$