Numerical Analysis

Math 370 Fall 1998 — MWF 11:30am - 12:25pm

 — Fowler 127

Class 6: **Wednesday September 16**

SUMMARY Orders of Convergence of a Sequence
CURRENT READING Burden & Faires (6th edition,) pages 36-37

Convergence of a Sequence

In a large number of numerical problems we will get a sequence of approximate answers to the single real number which is the exact solution of the problem we are looking at (e.g., a definite integral, a particular value of a solution to an initial value problem, etc). We often write this sequence as $x_1, x_2, x_3, \ldots$ and the limit as L or x_∞ and denote this by

$$\lim_{n \to \infty} x_n = L$$

The formal definition of the above limit can be written as:

We are interested in looking at **rates of convergence** of sequences.

Definition

Suppose we know that a sequence $\{\beta_n\}$ converges to zero and $\{\alpha_n\}$ converges to α.
The sequence $\{\alpha_n\}$ is said to converge to α at the **rate of convergence** $O(\beta_n)$ if

$$\lim_{n \to \infty} \frac{|\alpha_n|}{|\beta_n|} = K, \qquad \text{where } 0 < K < \infty$$

This is often written as $\alpha_n = \alpha + O(\beta_n)$

We read this as:

In addition, we say that $\{\alpha_n\}$ is $o(\beta_n)$ if

$$\lim_{n \to \infty} \frac{|\alpha_n|}{|\beta_n|} = 0$$

and that $\{\alpha_n\}$ is **equivalent** to $\{\beta_n\}$ (this is written $\alpha_n \sim \beta_n$) if

$$\lim_{n \to \infty} \frac{|\alpha_n|}{|\beta_n|} = 1$$

The typical orders of convergence that we are interested in are **linear**, **superlinear** and **quadratic** orders of convergence.

Can you write these orders using "big oh" and "little oh" notation?

Example
We use this knowledge about rates of convergence to compare competing algorithms. Suppose $\{A_n\}$ and $\{B_n\}$ are two algorithms described by

$$A_n = \frac{(n+1)}{n}, \qquad B_n = \frac{(n+3)}{n^2}$$

What does A_n converge to? What does B_n converge to?

Which of the sequences converges faster? (How can we find out?)

To help answer this question, try computing their **rates of convergence** relative to the sequences $\gamma_n = 1/n$ and $\tilde{\gamma_n} = 1/n^2$

GROUPWORK
Find the rates of convergence of the following sequences:

$$\lim_{n\to\infty} \frac{7n+2}{n^2} = \qquad \lim_{n\to\infty} \frac{n^2}{n^3} = \qquad \lim_{n\to\infty} \frac{n^3}{n^2} =$$

$$\lim_{n\to\infty} \frac{(7n+2)/n^2}{1/n^2} = \qquad \lim_{n\to\infty} \frac{(7n+2)/n^2}{1/n} = \qquad \lim_{n\to\infty} \frac{(7n+2)/n^2}{n} =$$

Rates of convergence

Here is a summary of the information about relative rates of convergence of sequences.

$$\lim_{n\to\infty} \frac{|\alpha_n|}{|\beta_n|} = \begin{cases} 0 & \text{means } \alpha_n = \alpha + o(\beta_n) \\ 1 & \text{means } \alpha_n \sim \beta_n \\ 0 < K < \infty & \text{means } \alpha_n = \alpha + O(\beta_n) \\ \infty & \text{means } \beta_n = \beta + o(\alpha_n) \end{cases}$$

Extension to Functions
The "big oh" and "little oh" notation can also be extended to describe the rate at which functions converge to values.

If we know that $\lim_{h\to 0} F(h) = L$ and and $\lim_{h\to 0} G(h) = 0$ If a positive constant K exists with

$$|F(h) - L| \le KG(h), \qquad mbox for sufficiently small h$$

then we write $F(h) = L + O(G(h))$

This can also be computed using the idea that $F(h) = L + O(G(h))$ if

$$\lim_{h\to 0} \frac{|F(h) - L|}{|G(h)|} = K$$

where K is some positive, finite constant.

Example
Show that $\cos(h) + \dfrac{h^2}{2} = 1 + O(h^4)$

What is the rate of convergence of $\sin(h^3)$ as $h \to 0$?