Numerical Analysis

Math 370 — MWF 2:30-3:25
Spring 1997 — 127 Fowler

(04.14.97) Evaluating Improper Integrals Numerically

Evaluating Improper Integrals Numerically

We have shown that we can make any kind of improper integral into an improper integral of the second kind where integrand becomes unbounded at the lower limit.
For the improper integral to CONVERGE, it must be able to written as

$$\int_a^b f(x)\,dx = \int_a^b \frac{g(x)}{(x-a)^p}\,dx, \qquad \text{for } 0 < p < 1$$

with $g(x)$ continuous on $[a,b]$

If $g \in C^5[a,b]$ we can construct a 4th degree Taylor Polynomial for $g(x)$ about the point $x = a$.

$P_4(x) =$

Therefore $\displaystyle\int_a^b f(x)\,dx = \int_a^b \frac{g(x) - P_4(x)}{(x-a)^p} + \frac{P_4(x)}{(x-a)^p}\,dx$

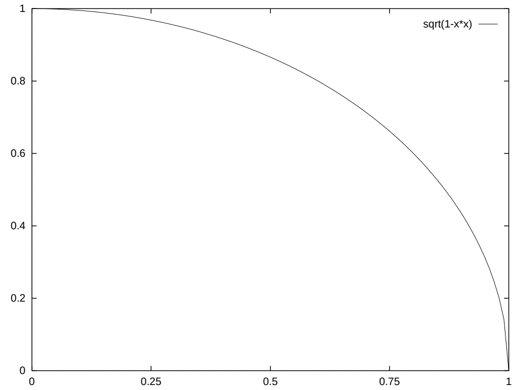
So, we end up having to evaluate two integrals,

$$\int_a^b G(x)\,dx \qquad \text{and} \quad \int_a^b \frac{P_4(x)}{(x-a)^p}\,dx$$

$$\text{where } G(x) = \begin{cases} \dfrac{g(x)}{(x-a)^p}, & \text{if } a < x \leq b \\ 0, & \text{if } x = a \end{cases}$$

The integral of $G(x)$ can be approximated very accurately using Simpson's Method (or Gaussian Quadrature or some other Precision 3 method) since it is $C^4[a,b]$, while the other integral is a simple convergent improper integral which can be computed by hand.

Example



x	$G(x)$
0.00	0.00
0.25	0.0000170
0.50	0.0004013
0.75	0.0026026
1.00	0.0099485

Find the value of $\displaystyle\int_0^1 \frac{e^x}{\sqrt{x}}\, dx$ numerically. You can use the data on $G(x)$ to help you.

Evaluating 2-D and 3-D integrals
Of course, these numerical methods can be applied to double and triple integrals as well. For example, let's use Gaussian Quadrature to evaluate
$\int_{-1}^{1}\int_{-1}^{1} x^2 + y^2 \, dx dy$

Let's evaluate the same integral using "basic" Simpson's Method

In general

$$\int_a^b \int_{c(x)}^{d(x)} f(x,y) dy dx \approx \int_a^b \frac{d(x)-c(x)}{2} \sum_{i=1}^{n} c_{n,i} f\left(x, \frac{(d(x)-c(x))x_{n,j} + d(x) + c(x)}{2}\right) dx$$

where $c_{n,i}$ are the weights for the n-point Gaussian quadrature rule and x_i are the roots of the n-th degree Legendre Polynomial, $P_n(x)$.