Numerical Analysis

Math 370 MWF 2:30-3:25
Spring 1997 127 Fowler

(04.11.97) Gaussian Quadrature and Improper Integrals

N	SAMPLE POINTS x_i	WEIGHTS c_i
2	$1/\sqrt{3}$	1
	$-1/\sqrt{3}$	1
3	$\sqrt{3/5}$	5/9
	0	8/9
	$-\sqrt{3/5}$	5/9

Example

Use 2-point Gaussian Quadrature to approximate $\int_{-1}^{1} 2x^2 dx$
What is the exact value? How accurate do we expect our answer to be?

Use 2-point Gaussian Quadrature to approximate $\int_{1}^{3} 1 - \frac{1}{x^2}\, dx$

So, in general, how can we transform the integral on [a,b] to an integral on [-1,1]?

$$\int_a^b f(x)dx = \int_{-1}^{1}$$

N-point Gaussian Quadrature

In general, the N-point Gaussian Quadrature rule will have precision $2N-1$. The theoretical error looks like

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{N} c_i f(x_i) + \frac{2(N!)^4}{(2N+1)[(2N)!]^3}(b-a)^{2N} f^{(2N)}(\xi)$$

Example

Compute the theoretical error bound for our approximation using $N=2$ Gaussian Quadrature to find $\displaystyle\int_1^3 1 - \frac{1}{x^2}\,dx$

Compare this theoretical error to the actual error.

Weights and Sample Points

It is clear that the x_1 and x_2 are the roots of the polynomial $x^2 - 1/3$. This is better known as the $N=2$ Legendre Polynomial. In fact, in general the x_i for the N-point Gaussian Quadrature Rule will be the roots of the N-th degree Legendre Polynomials.

The w_i are given by

$$w_i = \int_{-1}^{1} \prod_{\substack{j=1 \\ j\neq i}} \frac{x - x_j}{x_i - x_j}, \qquad \text{where } x_i \text{ are roots of } P_n(x)$$

Legendre Polynomials

The Legendre Polynomials P_n are a set of orthonormal polynomials on [-1,1]. This means that $P_N(x)$ is a polynomial of degree N and that $\displaystyle\int_{-1}^{1} P_k P_N dx = 0$ for all k less than N.

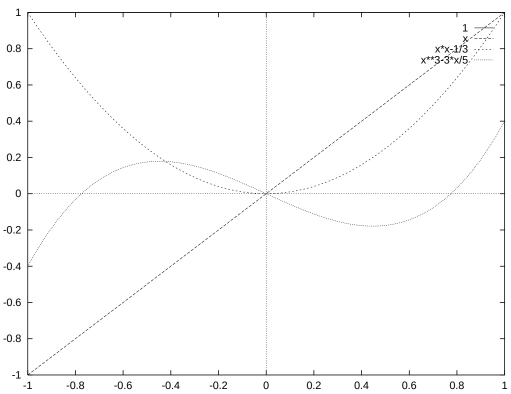
The Polynomials are determined recursively, using:

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x - \alpha_1 \\
P_{m+1}(x) &= xP_m - \alpha_{m+1}P_m(x) - \beta_{m+1}P_{m-1}(x)
\end{aligned}$$

where $\alpha_{m+1} = \dfrac{\int_{-1}^{1} xP_m^2 dx}{\int_{-1}^{1} P_m^2 dx}$ and $\beta_{m+1} = \dfrac{\int_{-1}^{1} xP_m P_{m-1} dx}{\int_{-1}^{1} P_{m-1}^2 dx}$

The first few Legendre Polynomials are

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= x^2 - 1/3 \\
P_3(x) &= x^3 - \frac{3}{5}x
\end{aligned}$$



Evaluating Improper Integrals Numerically

Recall from calculus that an integral is said to be improper when:

Note, that an improper integral of the first kind, $\int_a^\infty f(x)\,dx$ can be converted to an improper integral of the second kind by using the transformation $t = 1/x$. Show this.

So, if we can come up with a scheme for approximating improper integrals of the second kind then we can use this in many different situations.

For the improper integral of the second kind to CONVERGE, it must be able to written as

$$\int_a^b f(x)\,dx = \int_a^b \frac{g(x)}{(x-a)^p}\,dx, \qquad \text{for } 0 < p < 1$$

with $g(x)$ continuous on $[a, b]$

If $g \in C^5[a, b]$ we can construct a 4th degree Taylor Polynomial for $g(x)$ about the point $x = a$.

$P_4(x) =$

Therefore $\int_a^b f(x)\,dx = \int_a^b \frac{g(x) - P_4(x)}{(x-a)^p} + \frac{P_4(x)}{(x-a)^p}\,dx$

So, we end up having to evaluate two integrals,

$$\int_a^b G(x)\,dx \qquad \text{and} \quad \int_a^b \frac{P_4(x)}{(x-a)^p}\,dx$$

$$\text{where } G(x) = \begin{cases} \dfrac{g(x)}{(x-a)^p}, & \text{if } a < x \le b \\ 0, & \text{if } x = a \end{cases}$$

The integral of $G(x)$ can be approximated very accurately using Simpson's Method (or Gaussian Quadrature or some other Precision 3 method) since it is $C^4[a, b]$, while the other integral is a simple convergent improper integral which can be computed by hand.

NOTE that it doesn't matter that this improper integral of the second kind blows up at the left endpoint, we can always transform the problem into an associated problem where it blows up at the right endpoint. How do we do that?

Example

Find the value of $\int_{-1}^{0} \frac{e^x}{\sqrt{x}}\, dx$ numerically