Numerical Analysis

Math 370 | MWF 2:30-3:25
Spring 1997 | 127 Fowler

(04.09.97) Composite Numerical Integration; Gaussian Quadrature

Error Term of Composite Trapezoid

Starting with the formula for the Trapezoid Rule, we can show that the error formula for the Composite Trapezoid Rule is $O(\Delta x^2)$ and has precision 1.

$$\text{CT} = \text{T}(N, f, a, b) = \frac{\Delta x}{2}\left[f(a) + 2\sum_{j=1}^{N-1} f(a + j\Delta x) + f(b)\right] - \frac{(b-a)}{12}(\Delta x)^2 f''(\mu)$$

where $\mu \in (a, b)$ and $\Delta x = (b-a)/N$, N being number of subintervals

Composite Midpoint Method

$$\text{CM} = \text{M}(N, f, a, b) = \sum_{j=1}^{N} f[a + (j - 1/2)\Delta x]\Delta x + \frac{(b-a)}{24}(\Delta x)^2 f''(\mu)$$

Composite Simpson's Method

$$\text{CS} = \frac{2}{3}CM + \frac{1}{3}CT - \frac{(b-a)}{2880}(\Delta x)^4 f^{(4)}(\mu)$$

Example

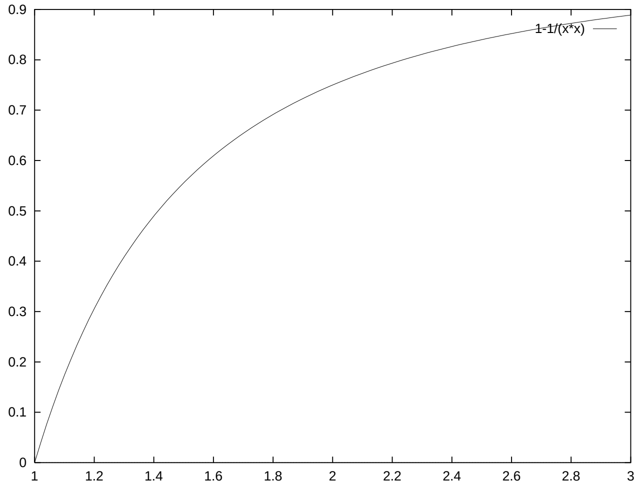
Now that we know expressions for the error terms of these methods we can use this information for **error control**. Consider $\int_1^3 e^{x^2}\,dx$.

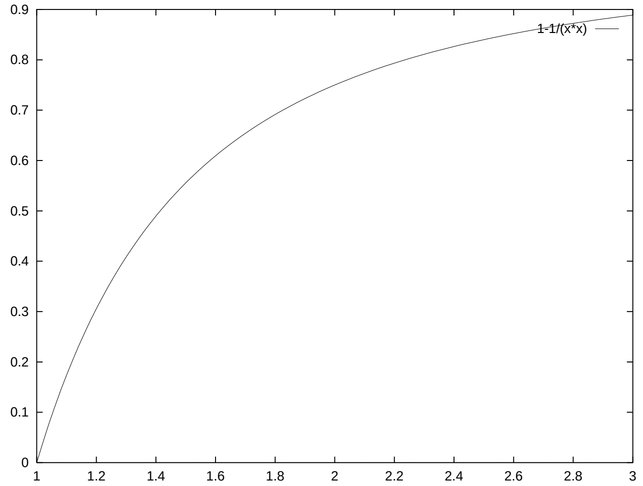
How many N subintervals would we have to split [1,3] into in order to be *confident* that you have computed the integral to with 10^{-4} of its exact value?

Compare how many subsintervals you need using Midpoint versus using Simpson's.

Effect of Round-Off Error on Composite Numerical Integration
Clearly when we do numerical integration we do quite a bit of function evaluation. As we saw in numerical differentiation, everytime we evaluate a function $f(x)$ the computer is using $\tilde{f}(x) + e(x)$. Lets consider the Trapezoid Method and try and compute $E(\Delta x)$ and see what happens as $\Delta x \to 0$

Gaussian Quadrature
Consider the following pictures of $f(x) = 1 - 1/x^2$. Using two function evaluations how would you estimate the area under the curve from 1 to 3?





Karl Friedrich Gauss had the idea of making the x_1 and x_2 variables as well as the coefficients c_1 and c_2. Then since this is 4 unknown parameters it should be able to approximate a polynomial which uses 4 unknown parameters to be completely described.
What degree polynomial is that?
Thus the $N = 2$ Gaussian Quadrature rule is **designed** to have precision ____________ .
Find x_1, x_2, c_1 and c_2 such that $\int_{-1}^{1} f(x)\,dx \approx c_1 f(x_1) + c_2 f(x_2)$
Let's write down the four equations for the four unknowns.

The x_i are known as the SAMPLE POINTS and the c_i are known as the WEIGHTS. Tables of these constants appear in many books. (See page 209 of Burden & Faires)

N	SAMPLE POINTS x_i	WEIGHTS c_i
2	$1/\sqrt{3}$	1
	$-1/\sqrt{3}$	1
3	$\sqrt{3/5}$	$5/9$
	0	$8/9$
	$-\sqrt{3/5}$	$5/9$

Example

Use 2-point Gaussian Quadrature to approximate $\int_{-1}^{1} 2x^2 dx$

What is the exact value? How accurate do we expect our answer to be?

Use 2-point Gaussian Quadrature to approximate $\int_{1}^{3} 1 - \frac{1}{x^2}\, dx$

So, in general, how can we transform the integral on [a,b] to an integral on [-1,1]?

$\int_{a}^{b} f(x)dx = \int_{-1}^{1}$ **N-point Gaussian Quadrature**

In general, the N-point Gaussian Quadrature rule will have precision $2N-1$. The formula will look like

$$\int_{-1}^{1} f(x)\, dx \approx \sum_{i=1}^{N} c_i f(x_i) + \frac{2^{2N+1}(N!)^4}{(2N+1)[(2N)!]^3} f^{2N}(\xi)$$

Weights and Sample Points

It is clear that the x_1 and x_2 are the roots of the polynomial $x^2 - 1/3$. This is better known as the $N = 2$ Legendre Polynomial. In fact, in general the x_i for the N-point Gaussian Quadrature Rule will be the roots of the N-th degree Legendre Polynomials.

The w_i are given by

$$w_i = \int_{-1}^{1} \prod_{\substack{j=1 \\ j \neq i}} \frac{x - x_j}{x_i - x_j}, \qquad \text{where } x_i \text{ are roots of } P_n(x)$$

Legendre Polynomials

The Legendre Polynomials P_n are a set of orthonormal polynomials on [-1,1]. This means that $P_N(x)$ is a polynomial of degree N and that $\int_{-1}^{1} P_k P_N dx = 0$ for all k less than N.

The Polynomials are determined recursively, using:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x - \alpha_1 \\ P_{m+1}(x) &= xP_m - \alpha_{m+1}P_m(x) - \beta_{m+1}P_{m-1}(x) \end{aligned}$$

where $\alpha_{m+1} = \dfrac{\int xP_m^2 dx}{\int P_m^2 dx}$ and $\beta_{m+1} = \dfrac{\int xP_m P_{m-1} dx}{\int P_{m-1}^2 dx}$

The first few Legendre Polynomials are

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= x^2 - 1/3 \\ P_3(x) &= x^3 - 3x/5 \end{aligned}$$