Numerical Analysis

Math 370 | MWF 2:30-3:25
Spring 1997 | 201 Fowler

(03.24.97) Applications of Polynomial Interpolation: NUMERICAL DIFFERENTIATION
We shall be using our knowledge of Lagrange Interpolation to come up with formulas that allow us to approximate the **derivative** of a function as a point, even though we are only given a number of values of the function.
From Calculus we know that if we know the function at $f(x_0)$ and $f(x_0 + h)$ then an approximation to the derivative of $f(x)$ at x_0, $f'(x_0)$ can be written as:

Example Consider the function $f(x) = \ln(x)$ at $x = 1.8$. In small groups of 2 or 3, obtain an approximation of the derivative, $f'(x)$, at $x = 1.8$ using different values of h less than 0.1. Also compute the actual error your approximation makes.

Do you see any relation between the choice of h and the size of the error?
What can we do if we want to get a bound on how large the actual error could get?

Write down the Lagrange interpolating polynomial for a function $f(x)$ given that you know the function at $(x_0, f(x_0))$ and $(x_1, f(x_1))$ (where $x_1 = x_0 + h$)

Write down the error term $e(x)$:

Let's write down an exact expression for $f(x)$ in terms of $e(x)$ and $P(x)$

and differentiate it:

Now, evaluate this expression for $f'(x)$ at $x = x_0$

and use the fact that $x_1 = x_0 + h$ to simplify this expression. Look familiar?

Suppose that the maximum value that f' has on $[x_0, x_0 + h]$ is M, then we can write down an error bound for our approximation to $f'(x_0)$

Example Compute theoretical error bounds for the approximations to $f'(1.8)$ that you had previously made. Compare the actual error to theoretical error bound. What do you see?

This particular kind of approximation for $f'(x_0)$ is called a **Forward Difference Formula**. If you replace h with $-h$ then you get the

Backward Difference Formula

$$f'(x_0) = \frac{f(x_0) - f(x_0 - h)}{h}$$

3-point Difference Formulae

What if instead of using 2 function values x_0 and $x_1 = x_0 + h$, we used THREE. That is, the function is evaluated at x_0, x_1 and x_2 and we use these to come up with an approximation to f'. Again we will use Lagrange polynomial interpolation for $f(x)$ along with the error term in order to try and approximate the derivative function $f'(x)$

If $x_1 = x_0 + h$ and $x_2 = x_0 + 2h$ this expression simplifies to:

If $x_1 = x_0 + h$ and $x_2 = x_0 - h$ write down the expression below:

This expression is called a **Centered Difference Formula**

Draw a sketch indicating how the various difference formulas approximate the derivative at a point. Do you see why they are approximations? What condition do we need on h to improve accuracy?

(n+1)-point Difference Formulae
In general, if one has $n+1$ data points for $f(x)$ at $x_0, x_1, \ldots x_n$ using Lagrange interpolation we can express the function as:

$$f(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

Differentiating produces...

$$\begin{aligned} f'(x) &= \sum_{k=0}^{n} f(x_k) L'_{n,k}(x) + \frac{d}{dx}\left[\frac{(x-x_0)(x-x_1)\ldots(x-x_n)}{(n+1)!}\right] f^{(n+1)}(\xi(x)) \\ &+ \frac{(x-x_0)(x-x_1)\ldots(x-x_n)}{(n+1)!}\frac{d}{dx}[f^{(n+1)}(\xi(x))] \end{aligned}$$

But if this evaluated at x_k then write down an expression for $f'(x_k)$ below:

Taylor Expansions
Another way to compute difference formula which approximate derivatives at a point is to use Taylor Approximations. This is useful because again one gets a sense of how good the approximation is as one is making it.
For example, let's write down the third-order Taylor expansion about x_0 (with error term) for the following expressions:
$f(x_0+h) =$ $f(x_0-h) =$ Then add these two terms and solve for $f'(x_0)$

This should result in the identical form of the 2nd Order centered difference formula for $f'(x)$ obtained by using a 3-point formula.