Numerical Analysis

Math 370 — MWF 2:30-3:25
Spring 1997 — 127 Fowler

(02.26.97) Polynomial Interpolation (continued)

Weierstrass Approximation Theorem

If f is defined and continuous on [a,b] and $\epsilon > 0$ is given, then there exists a polynomial $P(x)$, defined on [a,b] such that $|f(x) - P(x)| < \epsilon, \qquad$ for all $x \in [a,b]$.
What does this mean? Write out what this theorem means, in your own words.
Then share your meaning with your nearest neighbor.

Example

Recall the previous example of linear interpolation that we went through with U.S. Census Data. Consider the more general problem of having two data points (sometimes called **nodes**) (x_0, y_0) and (x_1, y_1). Write down a formula for the value of the interpolated value $P(x)$ below...

Lagrange Polynomials

The general form of the interpolating polynomial $P(x)$ of degree 1 through the points (x_0, y_0) and (x_1, y_1) has the form:

$$P(x) = \frac{(x - x_1)}{x_0 - x_1} y_0 + \frac{(x - x_0)}{(x_1 - x_0)} y_1$$

This can also be written as $P(x) = y_0 L_0(x) + y_1 L_1(x)$, where $L_k(x)$ are called **Lagrange Polynomials** of degree 1 for x_k. To be precise we should write these Lagrange polynomials as $L_{1,0}$ and $L_{1,1}$ to show that they are of degree 1 for x_0 and x_1, respectively.
Let's show that the answer you wrote down to the generic linear interpolation scheme is the same thing as the form when written using Lagrange Polynomials

Example

Show that $P(x_0)$ and $P(x_1)$ has the required values that you want.

What does this tell you about properties of $L_k(x_j)$? (Is there a difference whether $k = j$ or $k \neq j$)?

In general **Lagrange polynomials of degree** n, are given by

$$L_{n,k}(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}$$

Using a Lagrange Polynomial of degree n to interpolate a function value $P(x)$ between $n+1$ function values y_k produces

$$P(x) = \sum_{k=1}^{n} L_{n,k}(x) y_k$$

Error Formula for Lagrange Interpolation

The error between a function $f(x)$ and its interpolating polynomial $P(x)$ derived from the nodes $x_0, x_1, \ldots x_n$ on [a,b] is given by

$$e(x) = |f(x) - P(x)| = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1) \cdots (x - x_n)$$

where $\xi(x)$ is in (a, b)

Example

Determine the second degree interpolating polynomial for the function $f(x) = 1/x$ which uses the nodes $x_0 = 2, x_1 = 2.5$ and $x_2 = 4$

(a) First write down the ______ Lagrange polynomials of degree 2 and

(b) Combine your answer above to form $P(x)$

(c) Use your interpolant to approximate $P(3)$

NOTE: Wed and Fri Classes Are Held In Fowler 127