Numerical Analysis

Math 370 MWF 2:30-3:25
Spring 1997 127 Fowler

(02.10.97) Error Analysis of Iterative Methods
If we have a sequence of approximations $\{p_n\}$ which converges to p and there exist positive constants α and λ so that

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda = \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^\alpha}$$

then the sequence is said to **converge to p with order** α, with an **asymptotic error constant** λ. The method is said to be an "α order" method.

Recall that we define the error at each iterative step to be $e_n = p - p_n$

In other words the iterative method is said to be of order α if one can show a relationship like $|e_{n+1}| \approx \lambda |e_n|^\alpha$

Example
Consider the two sequences p_n and q_n which both to converge to zero.
However, p_n converges **linearly**, while q_n converges **quadratically**.

Let's assume that the two sequences have the same asymptotic error constant of $\lambda = 0.5$ and $p_0 = 1$. Use your calculator to generate the first 6 or so elements of each sequence and record these in the table below.

You should be able to come up with a formula which relates the n-th iterate to the initial guess (zeroth iterate) for both p_n and q_n and makes filling out the table easier....

Iterate n	Linear Convergence $p_n =$	Quadratic Convergence $q_n =$

Do you think it is a "big deal" whether a method is linearly convergent versus quadratically convergent?

Bisection Method
Recall the error formula for the bisection method and write it below:

Bisection is a _____ method, with $\lambda =$ _____ and $\alpha =$ _____
Fixed Point Iteration
We shall derive the order of convergence for Functional Iteration.
$p_{n+1} = g(p_n)$ and $e_{n+1} = p - p_{n+1}$ and $e_n = p - p_n$ therefor
$p_{n+1} = g(p - e_n) =$

Picard Iteration is a _____ method with $\alpha =$ _____
Newton's Method
In a similar fashion, we shall derive the order of convergence for Newton's Method

Method	Order of Convergence	Error Formula
Bisection		
False Position	1.442	
Secant	$\frac{1+\sqrt{5}}{2} \approx 1.618$	
Newton's		
Picard		

NOTE: The above only apply for simple roots (i.e. a root of multiplicity 1).
Definition
A root r of an equation $f(r) = 0$ has multiplicity m if and only if
$0 = f(r) = f'(r) = \cdots = f^{(m-1)}(r) = 0$ but $f^{(m)}(r) \neq 0$.

For roots of multiplicity $m > 1$ Newton's Method has the relationship that $|e_{n+1}| \approx \dfrac{M-1}{M}|e_n|$