Numerical Analysis

Math 370 — MWF 2:30-3:25
Spring 1997 — 127 Fowler

SUMMARY (01.24.97) Rates of Convergence and Orders of Magnitude

Let's go over this idea of measuring the rates at different sequences converge to their limit. First, let's go back to the concept of limit. Compute the following limits (You can use the program tt ORDER.TRU to assist you if you want)

$$\lim_{n\to\infty} \frac{7n+2}{n^2} = \qquad \lim_{n\to\infty} \frac{n^2}{n^3} = \qquad \lim_{n\to\infty} \frac{n^3}{n^2} =$$

$$\lim_{n\to\infty} \frac{(7n+2)/n^2}{1/n^2} = \qquad \lim_{n\to\infty} \frac{(7n+2)/n^2}{1/n} = \qquad \lim_{n\to\infty} \frac{(7n+2)/n^2}{n} =$$

Rates of convergence

Here is a summary of the information about relative rates of convergence of sequences.

$$\lim_{n\to\infty} \frac{|\alpha_n|}{|\beta_n|} = \begin{cases} 0 & \text{means } \alpha_n = \alpha + o(\beta_n) \\ 1 & \text{means } \alpha_n \sim \beta_n \\ 0 < K < \infty & \text{means } \alpha_n = \alpha + O(\beta_n) \\ \infty & \text{means } \beta_n = \beta + o(\alpha_n) \end{cases}$$

Orders of Magnitude

If α_n is $o(\beta_n)$ this means that α_n is of a **lower order of magnitude** than β_n
If α_n is $O(\beta_n)$ this means that α_n is the same **order of magnitude** as β_n

What's the relationship between relative orders of magnitude and relative rates of convergence?
Think about $\alpha_n = n^p$ and $\beta_n = n^q$. Derive a theory which relates orders of magnitude to rates of convergence which allows you to determine which sequence converges to its limit faster than the other. In other words answer the question, when is n^p tending to its limit faster than n^q?

Example

We use this knowledge about rates of convergence to compare competing sequences. Suppose $\{A_n\}$ and $\{B_n\}$ are two sequences described by

$$A_n = \frac{(n+1)}{n^2}, \qquad B_n = \frac{(n+3)}{n^3}$$

What does A_n converge to? What does B_n converge to?

Which of the sequences converges faster? (First we should ask "Which sequence has a higher order of magnitude?" Then we can say which is faster)

To help answer this question, try computing their **rates of convergence** relative to the sequences $\gamma_n = 1/n$ and $\tilde{\gamma_n} = 1/n^2$

Extension to Functions

The "big oh" and "little oh" notation can also be extended to describe the rate at which functions converge to values.

Write down your own definition of the statement $F(h) = L + O(G(h))$ below

Example

Show that $\cos h + h^2 = 1 + O(h^4)$

NOTE: Monday 1/27 Class Will Be In Fowler 201