Test 3: Numerical Analysis

Math 370 Friday April 18

Name:

Directions: Read **all** the problems first before answering any. You may use whatever books or notes that you wish.
You may **NOT** discuss the exam with other class members. If you have questions the only person you can can ask is Professor Buckmire by phone, by email or by coming to my office. This is an open book take-home exam.

Deadline: This take-home exam must be turned in by **Monday March 21 at 5:00pm**, IN FOWLER 402, my office.

No.	Score	Maximum
1		35
2		10
3		25
4		10
5		20
Total		100

1. *[35 pts. total]* **Numerical Integration.** Consider a new quadrature rule called **Buckmire's Rule**. It has the following form:

$$\int_{x_0}^{x_1} f(x)\,dx = w_0 f(x_0) + w_1 f(x_1) + w_2 f'(x_0) + w_3 f'(x_1) + K f^{(4)}(c)$$

where w_0, w_1, w_2, w_3 and K are unknown constants and $c \in (x_0, x_1)$.

Let $x_1 = x_0 + h$. You will derive the value of the four unknown constants w_i together with K in a similar manner to the method used to derive Simpson's Rule in Assessment #7. This involves assuming that the method integrates polynomials up to degree 3 exactly. Then solve four equations for the four unknown weights.

(a) *[14 pts]* If $f(x) = x^3$, write down an equation for the unknown constants by simplifying the left and right hand sides of the expression which results from applying Buckmire's Rule to the integral.

(b) *[3 pts]* Show that if you just consider the coefficients of the x_0 terms of your equation derived in part **(a)** one obtains the equation $3hw_1 + 6w_3 = h^2$

(c) *[4 pts]* Show that if you just consider the coefficients of $(x_0)^2$ of your equation derived in part **(a)** one obtains the equation $hw_1 + w_2 + w_3 = \dfrac{h^2}{2}$

(d) *[3 pts]* Show that if you just consider the coefficients of $(x_0)^3$ of your equation derived in part **(a)** one obtains the equation $w_0 + w_1 = h$

(e) *[3 pts]* Show that if you just consider all the terms that are left which do not contain x_0 in the equation derived in part **(a)** one obtains the equation $w_1 h + 3w_3 = \frac{h^2}{4}$

(f) *[8 pts]* Solve the system of 4 equations for 4 unknowns to obtain values for w_0, w_1, w_2 and w_3.

2. *[10 pts. total]*

(a) *[2 pts]* Why doesn't K appear in any of your equations in Question **(1)**?

(b) *[4 pts]* To find K, let $f(x) = x^4$ and repeat the process you used to obtain the last equation in part **1(e)** to get an equation for K. (HINT: Just compare the terms without x_0.)

(d) *[4 pts]* Solve the equation from **2(b)** to obtain K, so that you can write down Buckmire's Rule with all the unknown constants now known.

3. *[25 pts. total]* Previously you should have shown that Buckmire's Rule is given by:

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}f(x_0) + \frac{h}{2}f(x_1) + \frac{h^2}{12}f'(x_0) - \frac{h^2}{12}f'(x_1) + \frac{h^5}{720}f^{(4)}(c)$$

where $c \in (x_0, x_1)$

(a) *[5 pts]* Use Buckmire's rule to approximate $\displaystyle\int_1^\infty \frac{4}{1+x^2}\,dx$ (You may have to transform this integral).

(b) *[5 pts]* Use $N = 2$ Gaussian quadrature rule to approximate $\displaystyle\int_1^\infty \frac{4}{1+x^2}\,dx$

(c) *[5 pts]* Use the “basic” Trapezoid rule to approximate $\int_1^\infty \frac{4}{1+x^2}\,dx$

(d) *[10 pts]* Which of these methods is closest to the actual value of the integral? Why? (In other words, which do you *expect* to be the closest?)

4. *[10 pts. total]* Consider the "basic" Trapezoid Rule, $\dfrac{(b-a)}{2}f(a) + \dfrac{(b-a)}{2}f(b) - \dfrac{(b-a)^3}{12}f''(\xi)$, where $\xi \in (a,b)$.

(a) *[5 pts]* Suppose we guess that ξ is exactly at the midpoint, i.e. $\xi = \dfrac{a+b}{2}$. Write down a centered-difference approximation for $f''(\xi)$ in terms of $f'(a)$ and $f'(b)$ together with the error term for this approximation of f''

(b) *[5 pts]* Using your result in **(a)** to approximate the error term in the Trapezoid Rule do you see a relationship between the Trapezoid Rule and Buckmire's Rule? What is this relationship? How are they alike? How are they similar?

5. *[20 pts.]* **Numerical Differentiation.** After conducting an experiment we obtain the following data on some unknown function $f(x)$ and its unknown derivative $f'(x)$.

x_i	$f(x_i)$	$f'(x_i)$
0	0.9877659	0.0551344
1	0.9367786	-0.2466410

(a) *[4 pts.]* Use the given data to obtain approximations for $f'(0)$ and $f'(1)$

(b) *[2 pts.]* Compute the actual errors in the approximations you made in **(a)**

(c) *[4 pts.]* Use the given data to obtain approximations for $f''(0)$ and $f''(1)$

(d) *[2 pts.]* Use the given function data to obtain an approximation for $\int_0^1 f(x)\,dx$

(e) *[4 pts.]* Use all the given data to obtain another approximation for $\int_0^1 f(x)\,dx$

(f) *[4 pts.]* Which of these approximations you made in **(a)**, **(c)**, **(d)** and **(e)** do you think is the most accurate? **Why?** Explain your answer.