Numerical Analysis

Worksheet 16

SUMMARY Iterative Methods for Solving Systems of Linear Equations
READING Recktenwald, Sec 8.5, pp. 427–445; Sec. 7.1.2 and Sec 7.2.4

We have looked at methods for finding iterative solutions of systems of *nonlinear* equations. The methods we know are Newton's Methods for Systems (`newtonsys.m`), Successive Substitution (`succsub.m`) and Seidel Iteration (`seidel.m`).
Today we will consider using Iterative Methods to Solve Solutions of Linear Systems. Consider the system

$$\begin{aligned} 4x - y + z &= 7 \\ 4x - 8y + z &= -21 \\ -2x + y + 5z &= 15 \end{aligned}$$

We can show that the system has a unique solution : (2,4,3).
(How would you use MATLAB to find this solution?)

We can write this as a matrix equation $A\vec{x} = \vec{b}$ (**NOTE:** now A and b are not functions of $\vec{x}$)

$$\begin{pmatrix} 4 & -1 & 1 \\ 4 & -8 & 1 \\ -2 & 1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ -21 \\ 15 \end{pmatrix}$$

We can re-write these equations as

$$x = \frac{7 + y - z}{4}, \qquad y = \frac{21 + 4x + z}{8}, \qquad z = \frac{15 + 2x - y}{5}$$

This should suggest an iteration scheme $\vec{x}_{k+1} = \vec{G}(\vec{x}_k)$ Write it down below (in component form):

This scheme is called **Jacobi Iteration.**
Using your understanding of Seidel Iteration, you should be able to write down the iterative scheme which solves the system using **Gauss-Seidel Iteration**. Write that down below:

In general one can solve linear systems using iterative schemes of the form $\vec{x}_{k+1} = T\vec{x}_k + \vec{c}$ (where T depends on A and c depends on A and b)

GROUPWORK
Use the initial guess of $(1, 2, 2)^T$ and implement 2 steps of Gauss-Seidel Iteration and Jacobi Iteration to approximate the solution to the linear system. Which one gets closer to the actual solution of $(2, 4, 3)T$? How do you measure this "closeness"?

We could have also re-arranged the system as

$$\begin{aligned} -2x + y + 5z &= 15 \\ 4x - 8y + z &= -21 \\ 4x - y + z &= 7 \end{aligned}$$

$$\begin{pmatrix} -2 & 1 & 5 \\ 4 & -8 & 1 \\ 4 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 15 \\ -21 \\ 7 \end{pmatrix}$$

Use 2 steps of Gauss-Seidel **or** Jacobi iteration to approximate the solution of this system.

What happens to your approximation this time? Which of the versions of the linear system has a diagonally dominant matrix representation?

Diagonal Dominance
A matrix A of dimension N by N is said to be **strictly diagonally dominant** if and only if

$$|a_{kk}| > \sum_{\substack{j=1 \\ j \neq k}}^{N} |a_{kj}| \text{ for } k = 1, 2, \ldots, N$$

THEOREM
Suppose A is a strictly diagonally dominant matrix. Then $A\vec{x} = \vec{b}$ has a unique solution $\vec{x} = \vec{p}$. Jacobi Iteration and Gauss-Seidel Iteration will produce a sequence of vector $\vec{x}_n$ which will converge to $\vec{p}$ for any choice of the starting vector $\vec{p}_0$.
Note: Gauss-Seidel Iteration, when it converges, will converge faster than Jacobi Iteration, but there are some systems for which Jacobi Iteration will converge and Gauss-Seidel will diverge.

Norms

Given a vector $\vec{x}$ in $\mathcal{R}^n$ a norm is a function of $\vec{x}$ and produces a real number $||\vec{x}|| \in \mathcal{R}$. The Euclidean norm, sometimes denoted l_2 is the most common norm

$$||\vec{x}||_2 = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{x_1^2 + x_2^2 + x_3^2 + \ldots x_n^2}$$

Other popular norms are l_1 and l_∞

$$||\vec{x}||_1 = \sum_{i=1}^{n} |x_i| = |x_1| + |x_2| + |x_3| + \ldots |x_n|$$

$$||\vec{x}||_\infty = \max_{1 \leq i \leq n} |x_i|$$

Example

Given $\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ -4 \end{bmatrix}$, find $||\vec{x}||_1$, $||\vec{x}||_2$ and $||\vec{x}||_\infty$

The norm of a vector is, in general, a way to measure the "size of the vector." So, in iterative schemes, when we want to measure convergence, we look at the size of $||f(\underline{x}^{(k+1)})||$ or $||\underline{x}^{(k+1)} - \underline{x}^{(k)}||$ as a stopping criterion.

Note, in 1-dimension $||\vec{x}|| \equiv |x|$.

Note, that in 2-dimensions each norm has a very useful graphical interpretation. Draw a sketch of all the points $||\vec{x}|| \leq 1$ for l_1, l_2 and l_∞ on each of the 3 axes below.

Properties of Norms

1. $||\vec{x}|| \geq 0$ if and only if $\vec{x} \neq 0$

2. $||c\vec{x}|| = |c|||\vec{x}||$ for any scalar c

3. $||\vec{a} + \vec{b}|| \leq ||\vec{a}|| + ||\vec{b}||$ (The Triangle Inequality)

Matrix Norms
A matrix norm is similar to a vector norm in that it is a function which has a matrix as input and a real number (scalar) as an output.
The matrix norms we will be dealing with are the induced matrix norms l_1, l_2 and l_∞. The idea is that you are trying to find out how much the given matrix A can transform a unit vector $\underline{x}$.

$$||A||_2 = \max_{||\underline{x}||_2=1} ||A\underline{x}||_2$$

However, in practice the l_2 norm of a matrix is almost always computed by finding the spectral radius of the matrix: computing the square root of the largest eigenvalue of A^TA
Happily, the two "regular" norms of a matrix are very easy to compute for an $m \times n$ matrix A

$$||A||_1 = \max_{1\le j\le n} \sum_{i=1}^{m} |a_{ij}| \qquad \text{(max of the column sums)}$$

$$||A||_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}| \qquad \text{(max of the row sums)}$$

A funky norm...

$$||A||_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2} \qquad \text{Frobenius norm}$$

Example

Given $A = \begin{pmatrix} -2 & 1 & 5 \\ 4 & -8 & 1 \\ 4 & -1 & 1 \end{pmatrix}$ find $||A||_1$ and $||A||_\infty$. Use MATLAB to find $||A||_2$.

Extra Requirements on Matrix Norms
Matrix norms must obey the same basic properties of vector norms PLUS
4. $||AB|| \le ||A||||B||$
5. $||A\vec{x}|| \le ||A||||\vec{x}||$