Numerical Analysis

Math 370 Fall 2004 MWF 2:30 - 3:25pm
 Fowler North 5

Worksheet 4: **Monday September 13**

SUMMARY Taylor Series Approximations and Order of Convergence for Functions
CURRENT READING Recktenwald (Sec 5.3.1)

Order of Convergence of Functions
If we know that $\lim_{h\to 0} F(h) = L$ and $\lim_{h\to 0} G(h) = 0$ and if a positive constant K exists with

$$|F(h) - L| \cdot \quad KG(h), \qquad \text{for sufficiently small } h$$

then we write $F(h) = L + \mathcal{O}(G(h))$

This can also be computed using the idea that $F(h) = L + \mathcal{O}(G(h))$ if and only if

$$\lim_{h\to 0} \frac{|F(h) - L|}{|G(h)|} = K$$

where K is some positive, finite constant.
EXAMPLE
Show that the expression $\cos(h) + \frac{h^2}{2}$ is $1 + \mathcal{O}(h^4)$

What is the rate of convergence of $\sin(h^3)$ as $h \to 0$

Taylor Expansions

Another approach to figuring out order of convergence of functions is to use *Taylor Series Approximations* to assist you in satisfying the inequality version of the definition.

Recall that if you have a function $f(x)$ near a point $x = a$ and $f(x)$ is infinitely-differentiable, you can write down

$$f(x) =$$

or you can truncate this series and write down

$$f(x) \approx$$

Suppose we approximate the function $f(x)$ near the point $a = 0$ then we obtain what is known as a *Maclaurin Series.*

Maclaurin Series

$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(0)\frac{x^k}{k!}$$

If we are only interested in the behavior of the function for small values of x near 0, i.e. for $|h| << 1$ then we can write the expression as

$$f(0+h) = \sum_{k=0}^{\infty} f^{(k)}(0)\frac{h^k}{k!}$$

Interestingly, we can write an exact expression for the truncated form of this expression as

$$f(0+h) = f(0) + f'(0)h + f''(0)\frac{h^2}{2} + \mathcal{O}(h^3)$$

Yes, this last term is the same "big oh" that we have been discussing in regards to order of convergence of a function to its limit.

Exercise

Write down the following Taylor Series Approximations (for small h) for the following functions:

$\sin(h) \approx$

$\cos(h) \approx$

$e^h \approx$

$(1+h)^p \approx$

$\ln(1+h) \approx$

EXAMPLE

Show that you can use a truncated Maclaurin Expansion to prove that $\cos(h) + \frac{h^2}{2} = 1 + \mathcal{O}(h^4)$

The Three Ways Of Computing Order of Convergence of a Function are

1. **Limit Method**
2. **Bounding/Inequality Method**
3. **Truncated Taylor/Maclaurin Expansion**

(next to each of the methods above, write a short note to yourself explaining the method. Then share your notes with the student next to you.)

GROUPWORK

Work in Groups of 2 or 3 to find the limit and express the order of convergence in terms of $f(h) = c + \mathcal{O}(h^\alpha) = c + o(h^\beta)$ for the following:

1. $f(h) = \dfrac{1 + h - e^h}{h^2}$

2. $f(h) = \dfrac{1}{1 - h^4}$

Truncation Error

Whenever we approximate a continuous mathematical expression by a discrete algebraic expression we incur a truncation error. For example, we know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

If we do not actually evaluate the entire infinite series and stop at the 10000^{th} term we are making a truncation error. However, thanks to "big oh" notation we have a sense of how big this truncation error is.

$$e^x = 1 + x + \mathcal{O}(x^2)$$

In general, when we use a Taylor Polynomial of degree n to approximate a function $f(x)$ near some point $x = a$ we can write

$$f(x) = P_n(x,a) + \mathcal{O}((x-a)^n), \qquad \text{where } P_n(x,a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

The "exact form" of the truncation error when using Taylor Polynomials can be written as

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

where ξ is an unknown value. However you do know that $R_n(x) \to 0$ as $n \to \infty$

Example

Use second, fourth and sixth order Taylor Polynomials to approximate the value of $e = 2.718281828$. Find the absolute error in each case.