Test 1: Numerical Analysis

Math 370 Friday October 15, 2004

Name: ______________________

Directions: Read *ALL* three (3) problems first before answering any of them. This is a one hour, closed notes, closed book, test. This test has 5 pages. You must show all relevant work to support your answers. Use complete English sentences and indicate your final answer from your "scratch work."

No.	Score	Maximum
1		30
2		40
3		30
Total		100

1. *[30 points total.]* **Order Notation and Taylor Series.**

Consider the function $F(h) = e^{e^h - 1}$. Given that $F(h) = 1 + \mathcal{O}(h)$, our goal in this problem is to evaluate limits involving the function $F(h)$ as $h \to 0$.

(a) *[10 points].* Evaluate $\displaystyle\lim_{h\to 0^+} \frac{e^{e^h - 1} - 1}{1}$.

(b) *[10 points].* Evaluate $\displaystyle\lim_{h\to 0^+} \frac{e^{e^h - 1} - 1}{h}$.

(c) *[10 points].* $\displaystyle\lim_{h\to 0^+} \frac{e^{e^h - 1} - 1}{h^2}$.

[HINT: $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dots$]

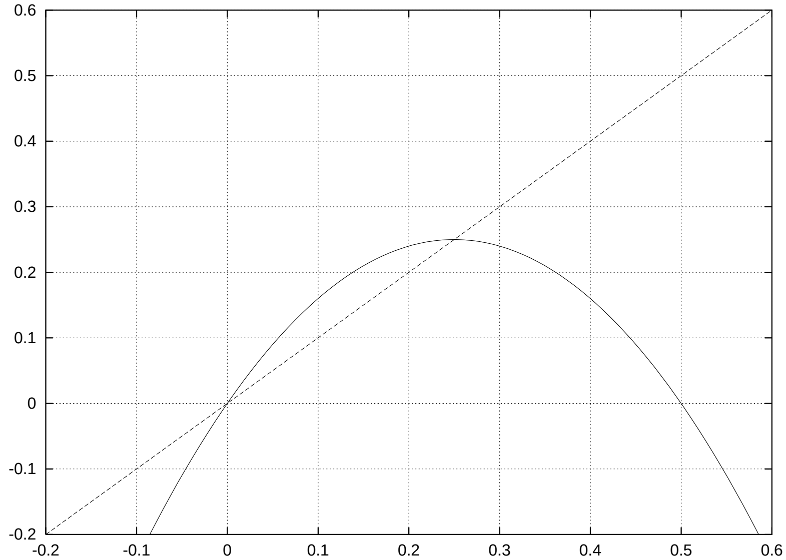
2. *[40 points total.]* **Sequences, Limits and Functional Iteration.**

Consider the functional iteration $x_{n+1} = g(x_n) = 2x_n(1-2x_n)$. Our goal in this problem is to completely describe what the limit x_∞ of this iteratively generated sequence will be depending on the starting value x_0.

(a) *[10 points].* For what values of x_0 will $x_0 = x_1 = x_2 = \ldots = x_\infty$? [In other words, find the two fixed points of $g(x)$.]

(b) *[15 points].* **Show experimentally that one of these special values of x_0 in part (a) is an attractive fixed point and one is a repulsive fixed point.** [An attractive fixed point x^* is one in which there exists a $\delta > 0$ so that every sequence starting from an x_0 in the interval $(x^* - \delta, x^* + \delta)$ converges to x^*. A repulsive fixed point x^* is one in which there exists a $\delta > 0$ so that the only sequence starting from an x_0 in an interval $(x^* - \delta, x + \delta)$ that converges to x^* is the one where $x_0 = x^*$.]

Here is a graph of $y = g(x) = 2x(1 - 2x)$ and $y = x$ on the interval $-0.2 \le x \le 0.6$.



(c) *[5 points].* Despite one of the fixed points being a repulsive fixed point there exists another point on the real number line for which if $x_0 \neq x^*$ the sequence $x_{n+1} = g(x_n) = 2x_n(1 - 2x_n)$ will still converge to this repulsive point. Find this value x_0.

(d) *[10 points].* Draw a real number line or table below and clearly indicate for which initial values x_0 the sequence will converge to a finite value x_∞ and for which initial values x_0 the sequence will diverge to an infinite value. In other words, make sure you are defining a function which takes any real number x_0 as input and produces x_∞ as output. **Is it possible for the sequence to converge to $+\infty$?**

3. *[30 points total.]* **TRUE or FALSE.**

Are the following statements TRUE or FALSE – put your answer in the box. To receive ANY credit, you must also give a brief, and correct, explanation in support of your answer! For example, if you think the answer is **FALSE** providing a counter example for which the statement is NOT true is best. If you think the answer is **TRUE** you should also explain why you believe the statement. Your explanation of your answer is worth FOUR TIMES as much as the answer you put in the box.

(a) Newton's Method will always converge faster to the solution of $f(x) = 0$ than the Bisection Method will (assuming Bisection is given a bracket containing the root and Newton's initial guess IS "close enough" to the solution).

(b) MATLAB can represent (and do calculations with) non-zero real numbers closer to zero than `realmin=2.25e-308`.

(c) If two convergent sequences, p_n and q_n are both linearly convergent then they take the same number of steps to be within ϵ of their respective limits p_∞ and q_∞.