Numerical Analysis Final Project

The Bratu Problem

Version: November 29, 2004 Due Date: **Monday December 13, 5:00pm**

1 Background

The nonlinear eigenvalue problem $\Delta u + \lambda e^u = 0$ in the unit square with $u = 0$ on the boundary is often referred to as "the classical Bratu problem" or "Bratu's problem." The Bratu problem in 1-dimensional planar coordinates, $u'' + \lambda e^u = 0$ with $u(0) = u(1) = 0$ has two known, bifurcated, exact solutions for values of $\lambda < \lambda_c$ and no solutions for $\lambda > \lambda_c$. The value of λ_c is simply $8(\alpha^2 - 1)$ where α is the fixed point of the hyperbolic cotangent function $\coth(x)$. In this project, numerical approximations to the exact solution of the one-dimensional planar Bratu problem will be computed using various numerical methods. Of particular interest will be the application of nonstandard finite-difference schemes known as Mickens finite differences to solve the problem. For extra credit, these techniques can be applied to the classic Bratu problem in the unit square.

Our goal is to produce numerically accurate solutions to the 1-D Bratu problem using consistent numerical approximations. We shall consider a problem "numerically accurate" when the difference between our computed solution and the exact solution is less than some given tolerance. A consistent numerical appoximation is one in which as the level of discretization increases, the error goes to zero. In the case of the classic Bratu problem there is no known exact solution, so we will consider the problem solved when the difference between consecutive computed solution is less than some given tolerance.

2 The Planar Bratu Problem(s)

The classical Bratu problem is

$$\Delta u + \lambda e^u = 0 \qquad \text{on } \Omega : \{(x, y) \in 0 \le x \le 1, 0 \le y \le 1\} \tag{1}$$

$$\text{with } u = 0 \qquad \text{on } \partial\Omega \tag{2}$$

where u is a function of x and y, λ is an unknown constant (eigenvalue), Δ is the Laplacian operator equal to $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ on Ω which is the unit square with its lower left corner at the origin. The boundary condition is that $u(x, y)$ is zero on the perimeter or boundary of the square, denoted by $\partial\Omega$. This version of the Bratu Problem can be referred to as the 2-D Planar Bratu Problem. There is no known explicit exact solution to the 2-D Planar Bratu Problem.

The 1-dimensional version of this problem is called the 1-D Planar Bratu Problem and is

$$u''(x) + \lambda e^{u(x)} = 0 \qquad 0 \le x \le 1, \tag{3}$$

$$\text{with } u(0) = 0 \qquad \text{and } u(1) = 0 \tag{4}$$

The exact solution to (3) is known and can be presented here as

$$u(x) = -2 \ln \left[\frac{\cosh((x - \frac{1}{2})\frac{\theta}{2})}{\cosh(\frac{\theta}{4})} \right] \tag{5}$$

where θ solves

$$\theta = \sqrt{2\lambda}\cosh\left(\frac{\theta}{4}\right). \tag{6}$$

There are two solutions to (6) for values of $0 < \lambda < \lambda_c$. For $\lambda > \lambda_c$ there are no solutions. The solution (5) is only unique for a critical value of $\lambda = \lambda_c$ which solves

$$1 = \sqrt{2\lambda_c}\sinh\left(\frac{\theta_c}{4}\right)\frac{1}{4}. \tag{7}$$

By graphing the line $y = \theta$ and the curve $y = \sqrt{2\lambda}\cosh(\frac{\theta}{4})$ for fixed values of $\lambda = 1, 2, 3, 4$ and 5 the solutions of (6) can be seen as the points of intersections of the curve and the line in Figure 1. Clearly, there is only one solution when the $y = \theta$ line is exactly tangential to the $y = \sqrt{2\lambda}\cosh(\frac{\theta}{4})$ curve, which leads to the condition given in (7).

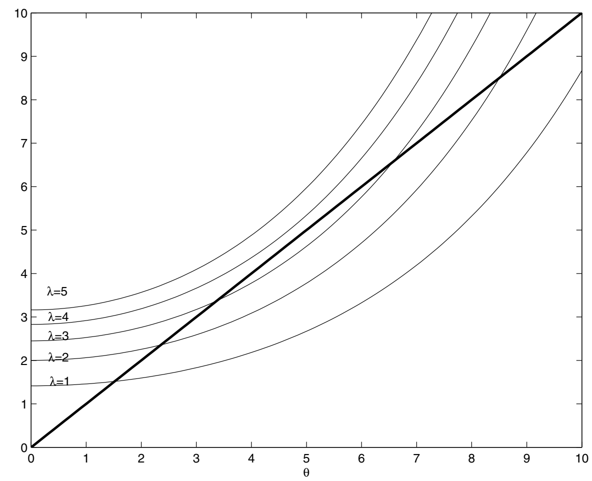

Figure 1: Graphical depiction of dependence of solutions of (6) upon λ

Dividing (7) by (6) produces:

$$\begin{aligned}\frac{4}{\theta_c} &= \tanh\left(\frac{\theta_c}{4}\right)\\ \Rightarrow \frac{\theta_c}{4} &= \coth\left(\frac{\theta_c}{4}\right)\\ \Rightarrow \alpha &= \coth(\alpha)\end{aligned}$$

The critical value θ_c is four times α, which is the positive fixed point of the hyperbolic cotangent function, 1.19967864.

$$\theta_c = 4.79871456 \tag{8}$$

The exact value of θ_c can therefore be used in (7) to obtain the exact value of λ_c.

$$\lambda_c = \frac{8}{\sinh^2\left(\frac{\theta_c}{4}\right)} = 8(\alpha^2 - 1) \Rightarrow \lambda_c = 3.513830719 \tag{9}$$

The relationship between λ and θ for some values of λ less than λ_c are given in Table 1. If you want to know the values of θ corresponding to other values of λ you could interpolate between values in the table. Obviously, when $\lambda = \lambda_c$ then $\theta_1 = \theta_2 = \theta_c$ and when $\lambda > \lambda_c$ there are no solutions to (6). Also, when $0 < \lambda < \lambda_c$ there are two solutions to (6). This

λ	θ_1	θ_2
0.5	1.0356946	13.0382393
1.0	1.5171645	10.9387028
1.5	1.9397652	9.5816998
2.0	2.3575510	8.5071995
2.5	2.8115549	7.5480981
3.0	3.3735077	6.5765692
3.5	4.5518536	5.0543427
λ_c	4.7987146	4.7987146

Table 1: Corresponding values of θ for various $\lambda \leq \lambda_c$

means there are two valid functions $u(x)$ which solve (3) for every value of λ below the critical value λ_c.

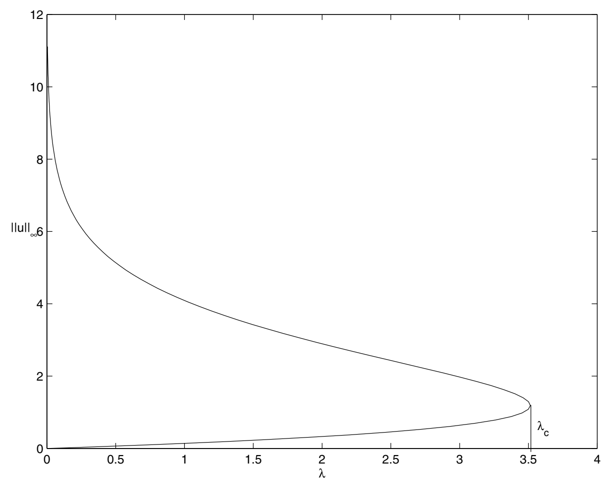

Figure 2: Bifurcated nature of the exact solution to the Bratu problem

Figure 2 shows how the maximum value of the solution function (5) depends on the nonlinear eigenvalue λ with the critical value of λ_c highlighted at the "turning point." Table 1 and Figure 2 are two different ways of depicting the property of the solution that it is double-valued for $\lambda < \lambda_c$. In the next section, numerical methods to compute these solutions to (3) will be discussed.

3 Numerical Methods

In this section of the project, the details of the numerical methods used to compute solutions to (3) shall be given. The method involves approximating the differential equation with standard finite differences and using Newton's Method to solve the resulting nonlinear system of equations. Both standard and nonstandard (Mickens) finite-difference schemes can be used to approximate derivatives.

Finite Difference Methods

To solve a boundary value problem using finite differences involves discretizing the differential equation and boundary conditions. This method transforms the problem into a system of simultaneous nonlinear equations which are then usually easily solved using Newton's method. There are many choices for how to approximate the derivatives which appear in a differential equation. In this section of the paper standard finite differences and nonstandard finite differences will be deployed. Nonstandard finite differences have been extensively studied by Professor Ronald E. Mickens of Clark Atlanta University. The first step in the computation of the numerical solution of (3) using a finite-difference method is to approximate the continuous domain of the problem with a discrete grid. The grid chosen was $\{x_j\}_{j=0}^{N}$ on the interval $0 \leq x \leq 1$ where

$$0 = x_0 < x_1 < x_2 < \ldots < x_j < \ldots < x_N = 1. \tag{10}$$

For a uniform grid, the grid separation parameter h is constant and $h = 1/N$ with $x_k = 0 + kh$ for $k = 0$ to N. Using a standard finite-difference scheme, the discrete version of the 1-D planar Bratu problem (3) will be

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + \lambda e^{u_j} = 0, \qquad j = 1, 2, \ldots, N-1 \tag{11}$$

with discrete boundary conditions

$$u_0 = 0 \qquad \text{and } u_N = 0. \tag{12}$$

The above equation (11) is known as an ordinary difference equation, or OΔE. The values u_j correspond to the unknown solution function $u(x)$ being evaluated at the discrete grid points x_j, in other words, they are approximations to $u(x_j)$. The hope is that the exact solution of the ordinary differential equation (ODE) evaluated at these discrete points is close to the exact solution of the discrete version of the differential equation (OΔE). Another way to say this is to talk about the discrete exact solution $u(x_j)$ and the exact discrete solution u_j. If the numerical solution technique for approximating the differential equation and solving the approximate problem is a good one, these two values should have a relationship. This is the main idea that you will be confirming in this Project for a number of different scenarios or examples.

3.1 N=2 Example

Consider the case where the grid is split into just 2 equal sub-intervals, i.e. $N = 2$. Then there are $N+1 = 3$ grid points, x_0, x_1 and x_2 and three corresponding unknown function values u_0, u_1 and u_2. However, by applying the boundary conditions (12) we know that $u_0 = 0$ and $u_2 = 0$. By applying these conditions to (11) we also know that there is only $N - 2 = 1$ unknown value, u_1, which solves the equation

$$\frac{-2u_1}{h^2} + \lambda e^{u_1} = 0 \tag{13}$$

where $h = 1/2$.

QUESTION: Using $\lambda = \lambda_c$ solve the above equation for u_1 and compare it to the value the exact solution in (5) has at $x_1 = 0.5$. *Note: this is a scalar root-finding problem*

3.2 N=3 Example

Consider the case where the grid is split into just 3 equal sub-intervals, i.e. $N = 3$. Now there are $N+1 = 4$ grid points, x_0, x_1, x_2 and x_3. You can show that there are now only

$N-2=2$ unknowns, u_1 and u_2 which solve the equations:

$$\begin{aligned} \frac{u_2 - 2u_1}{h^2} + \lambda e^{u_1} &= 0 \\ \frac{-2u_2 + u_1}{h^2} + \lambda e^{u_2} &= 0 \end{aligned} \qquad (14)$$

since $u_0 = u_3 = 0$ and $h = 1/3$. *Note: this is a vector root-finding problem*

QUESTION: Using $\lambda = \lambda_c$ solve the above equation (14) for u_1 and u_2 and compare it to the value the exact solution in (5) has at $x_1 = 1/3$ and $x_2 = 2/3$. What do you notice about the relationship between the values of u_1 and u_2?

3.3 N=10, 20, 40, 50, 100, 200, 500, 1000, ... Example

As the number of subintervals N goes to infinity, the discrete grid begins to resemble the continuous real line, so we expect the difference between the discrete exact solution and the exact discrete solution to go to zero. If we define an error vector whose k^{th} component is $e_k = |u(x_k) - u_k|$ then the norm of this error vector should decrease ($||\vec{e}|| \to 0$) as the number of subintervals increases ($N \to \infty$). The goal in this section is to get a sense of how this happens, in other words, quantify the relationship between the error E and the number of subintervals N (or equivalently, the grid separation parameter $h = 1/N$). Assuming that the relationship looks like $E = Ch^p$ then $E = 0 + \mathcal{O}(h^p)$. Use Approximation Theory to find the curve of least square error which fits best to the data you have. You should produce a table which has a column with N, a column with h, a column with E. You can use more than the requested values of $N = 2, 3, 10, 20, 40, 50, 100, 200, 500, 1000$.

QUESTION: Produce an m-file which takes as input: N, the vector function $\vec{f}$, its Jacobian, an error tolerance; and produces as output: the converged vector solution, the estimated error.

QUESTION: Produce a graph of E versus h which allows you to obtain a value for p and thus make a statement about how the error E goes to zero: linearly, superlinearly, quadratically or something else?

3.4 Nonstandard Finite Difference Example

A nonstandard finite-difference scheme for (3) is

$$\frac{u_{j+1} - 2u_j + u_{j-1}}{2\ln[\cosh(h)]} + \lambda e^{u_j} = 0, \qquad j = 1, 2, \ldots, N-1 \qquad (15)$$

The discretization given in (15) is an example of a Mickens discretization. Mickens has repeatedly shown that one can find nonstandard finite difference schemes which produce *exact* discrete solutions of a differential equation [3]. For example, in [4] the following Mickens scheme

$$\frac{u_{j+1} - u_j}{\left(\dfrac{1 - e^{-\alpha h}}{\alpha}\right)} = -\alpha u_j \qquad (16)$$

is an exact nonstandard finite difference scheme for the differential equation $\dfrac{du}{dx} = -\alpha u$. Also found in [4] is the following exact Mickens discretization for $\dfrac{du}{dx} = -u^3$.

$$\frac{u_{j+1} - u_j}{h} = -\left(\frac{2u_{j+1}}{u_{j+1} + u_j}\right) u_{j+1} u_j^2 \qquad (17)$$

A Mickens difference is a nonstandard finite-difference scheme which (1) approximates a derivative using a nonlinear denominator function and/or (2) uses "non-local" or "off-grid" representations of expressions in the differential equation.

The scheme given in (16) is an example of the use of a nonlinear denominator function in a Mickens finite difference. Note that the denominator function in (16), $\phi(h) = \dfrac{1 - e^{-\alpha h}}{\alpha}$ has the property that in the limit as $h \to 0$, $\phi(h) \to h$. In general, the denominator function ϕ in a Mickens finite-difference for the first derivative

$$u' \approx \frac{u_{j+1} - u_j}{\phi(h)}$$

has the property that $\phi(h) = h + o(h)$.

The scheme given in (17) is an example of a "non-local" discretization appearing in a Mickens difference. The standard discrete representation of u^3 would be expected to be simply u_j^3. However the unexpectedly florid discretization of this cubic term that appears on the right-hand side of (17) leads to an *exact* discrete solution to the differential equation.

The nonstandard finite difference scheme given in (15) is a Mickens difference for a second derivative

$$u'' \approx \frac{u_{j+1} - 2u_j + u_{j-1}}{\phi(h)}$$

where the denominator function $\phi(h) = 2\ln[\cosh(h)] = h^2 + o(h^2)$. Thus, in the limit as $h \to 0$ the standard finite-difference scheme (11) and the Mickens-difference scheme (15) will be identical. However, for the finite values of h at which numerical computations are conducted the hypothesis is that the nonstandard form of the denominator function $\phi(h)$ will lead to improved accuracy.

Using $\lambda = \lambda_c$ and the Mickens discretization given in (15) try to obtain the relationship between the error E and h, the grid separation parameter by solving the system for the same N values as in Section 3.3.

4 Extra Credit

The Extra Credit problem involves solving the classic Bratu problem, i.e. the 2-D Planar Bratu Problem using standard discretizations and Newton's Method for Systems. In this problem you have to discretize the unit square, so the exact discrete solution is denoted $u_{i,j}$ and the discrete exact solution is $u(x_i, y_j)$ where $x_i = 0 + i/N$ and $i = 0, 1, \dots, N$ and $y_j = 0 + j/N$ and $j = 0, 1, \dots, N$. In this case, the smallest value of N which makes sense of $N = 2$ which gives you $(N+1)^2 = 9$ grid points, though only one where the exact discrete solution is non-zero: $u_{1,1}$. Repeat the questions from Section 3, this time, in 2-dimensions.

Assignment

Solving the Bratu Problem

1. Show that the given function $u(x)$ in (5) is an exact solution of the boundary value problem for the 1-D Bratu Problem given that θ satisfies (6).

2. Confirm the given values of θ_c and λ_c through computation

3. Produce an m-file which given a value of λ as input, outputs the two corresponding values of θ

4. Create a graph with both solutions to the 1-D Bratu Problem for $\lambda = 1.25$

5. Solve the N=2 version of the 1-D Bratu Problem (using Newton's Method) and compare to the exact solution for $\lambda = \lambda_c$

6. Solve the N=3 version of the 1-D Bratu Problem (using Newton's Method for Systems) and compare to the exact solution for $\lambda = \lambda_c$

7. Write down the system of nonlinear equations with $\lambda = \lambda_c$ for the discretized version of the 1-D Bratu Problem for any N, including the corresponding Jacobian matrix

Numerical Methods

1. Create an m-file which will solve the system of nonlinear equations corresponding to discretizing the grid into N equal subintervals and compare this numerical solution to the exact solution

2. Produce a table of values obtained by solving the 1-D Bratu Problem for different increasing values of N which reflect the error E and the grid parameter h obtained from using standard finite differences

3. Use Approximation Theory to find the curve of least square error of the form $E = Ch^p$ using standard finite differences, particularly the value of p

4. Produce a table of values obtained by solving the 1-D Bratu Problem for different increasing values of N which reflect the error E and the grid parameter h obtained from using Mickens finite differences

5. Use Approximation Theory to find the curve of least square error of the form $E = Ch^p$ using Mickenss finite differences, particularly the value of p

6. Discuss which finite difference technique has the more favorable error profile

EXTRA CREDIT

1. Write down the system of nonlinear equations for the discretized version of the 2-D Bratu Problem for any N, including the corresponding Jacobian matrix

2. Solve the system of nonlinear equations for using a discretization of $N > 50$

References

[1] Bratu, G. "Sur les équations intégrales non-linéaires." *Bulletins of the Mathematical Society of France.* Volume **42**. 1914. pp. 113-142.

[2] Buckmire, R. "On exact and numerical solutions of the one-dimensional planar Bratu problem." Preprint. Available at http://faculty.oxy.edu/ron/research/.

[3] Mickens, R.E. "Difference equation models of differential equations having zero local truncation errors." *Differential equations.*(Birmingham, Ala., 1983), 445–449, North-Holland Math. Stud., 92, North-Holland: Amsterdam-New York, 1984.

[4] Mickens, R.E. "Difference Equation Models of Differential Equations." *Mathl. Comput. Modelling.* Volume **11**. 1988. pp. 528-530.

Report

Write a concise report containing the following sections.

1. **Problem Overview**: A brief statement of the project objective and a summary of the steps you used to achieve it.

2. **Mathematical Formulation**: Summarize the equations used in your analysis. Describe each variable in words. Be sure to identify the role of each equation in the overall analysis.

3. **Program Listings**: You should produce m-files The purpose of each `m-file` should be stated in the text of your report. Code listings, especially those that span multiple pages, should appear in an Appendix. The input and output variables for the modules need not be described separately as long as they are adequately documented in the function prologue.

4. **Results and Discussion**: Provide answers to the questions posed in the *Assignment* section, above. Your report need not following the numbering convention in the *Assignment* so long as all the issues raised there are discussed.

5. **Feedback on Group Dynamics**: Provide a summary of how your group worked together, summarizing how many meetings occurred, how long they lasted, who was responsible for which sections of the project, et cetera. This could be done through separate paragraphs, authored by each group member.

6. **Conclusion**: In one crisp paragraph, summarize the results of this project. *Do not present new information in the Conclusion.*

The report is to be delivered in hard copy by 5:00 PM on the due date for the project. The m-files must be emailed to me by this deadline.

Submission of Code

In addition to a hardcopy of the written report, the final working version of your MATLAB programs, along with basic instructions for running them, are to be included in email. The instructions for each program should be contained in the email with the corresponding attached m-file. The instructions should briefly (one or two sentences should do) describe how to run your code. Be sure to specify any input parameters that may be needed. When I run your code(s) I should be able to recreate all the results in your report.

Grading Criteria

The following criteria will be used to grade the final project

Category		Points
Technical content		
Check of exact solution to 1-D Bratu Problem		5
Graphs of both solutions when $\lambda = 1.25$		10
Setting Up Newton's Method to Solve 1-D Bratu Problem for any N		20
m-file which computes ($\vec{f}$, Jacobian) for Newton's Method		5
Numerical Results		
Accuracy of θ_c and λ_c values		5
Plot of θ versus λ		5
Tabulated Results for $N = 2, 3, 10, 20, 40, 50, 100, 1000$ using Mickens and Standard		20
Estimate of p from E versus h graphs using Standard FD to solve		5
Estimate of p from E versus h graphs using Mickens to solve		5
Documentation		
Organization and documentation of m-files		5
Discussion of group dynamics		5
Grammar, style, spelling		10
	Total	100
Extra Credit: Classic Bratu Problem in 2-D		
Setting Up The Numerical Solution		20
Producing "Converged" Numerical Solution for $N > 50$		20
Surface Plots of the Numerical Solution		10
	Extra Credit Total	50

Report Style

The following items fall under the category of "style."

- The report should be organized into major sections.
- The text should be written in complete sentences. It should be free of slang. All abbreviations and acronyms should be defined.
- Figures must have captions. Axes must have labels. Figures and tables of results may be placed at the end of the text body, but should not be placed in an appendix. All figures and tables of results that are not discussed in the body of the text will be ignored.
- Pages in your report should be numbered.
- Only items of secondary importance are put in an appendix.

To simplify your report, assume that the reader

- is familiar with the finite differences,
- is a competent MATLAB user,
- is unimpressed by fancy report covers,

Do not assume that the reader has a copy of the assignment sheet. This requires, for example, that you define all variables and constants that appear in any equations you present.