Differential Equations

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Worksheet 17

TITLE Straight Line Solutions
CURRENT READING Blanchard, 3.2

Homework #6 due Friday October 17 (* indicates EXTRA CREDIT)
Section 3.1: 6, 7, 8, 10, 13, 18*.
Section 3.2: 8, 9, 12, 16*, 17, 18*.

SUMMARY
Eigenvalues and eigenvectors return from Linear Algebra and are important in the case where Linear Systems of ODEs have solutions that look like straight lines.

1. The Significance of Eigenvectors and Eigenvalues

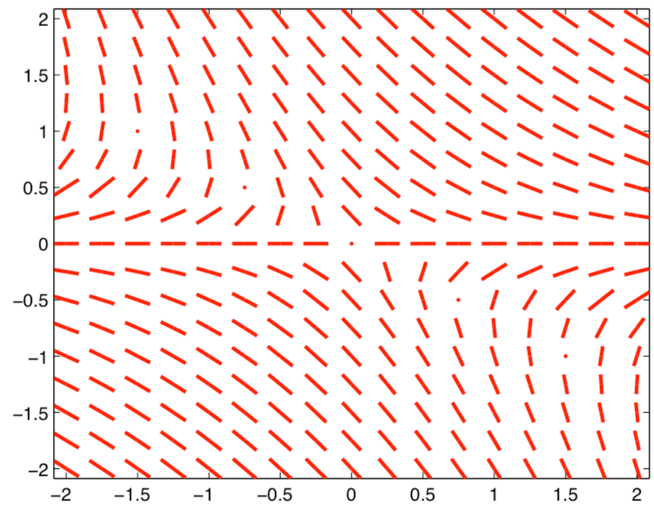
Recall the solutions $\vec{x}_1(t) = \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}$ and $\vec{x}_2(t) = \begin{bmatrix} -e^{-4t} \\ 2e^{-4t} \end{bmatrix}$ to the ODE $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix}\vec{x}$ from *Worksheet #16*.

Notice that $\vec{x}_1(t) = \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t}$ and $\vec{x}_2(t) = \begin{bmatrix} -e^{-4t} \\ 2e^{-4t} \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-4t}$.

Question
Do you notice anything interesting about the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$? Any relationship to the matrix $A = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix}$? What happens if you multiply each vector by A?

Answer
The vectors in question are ______________________________.

Consider the direction field for the ODE $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix} \vec{x}$:



It turns out that the general solution to $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix} \vec{x}$ can be written as
$\vec{x} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-4t}$.

Exercise

On the above direction field, we want to draw in the solutions $\vec{x}_1(t)$ and $\vec{x}_2(t)$. Does it matter what your initial condition is?

What happens as $t \to \infty$? What about as $t \to -\infty$ (i.e. reverse direction of the arrows)? Does one of the solutions seem more "attractive" than the other?

EXAMPLE

Consider the system $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \vec{x}$. Find the eigenvalues λ and eigenvectors $\vec{v}$ of $\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}$.

Show that the general solution can be written as $\vec{x} = c_1\vec{v}_1 e^{\lambda_1 t} + c_1\vec{v}_2 e^{\lambda_2 t}$ and confirm that it is actually a solution of $\vec{x}' = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \vec{x}$.

2. General Solution To Homogeneous Linear Systems

THEOREM

The general solution $\vec{x}(t)$ on the interval $(-\infty, \infty)$ to a homogeneous system of linear DEs $\dfrac{d\vec{x}(t)}{dt} = A(t)\vec{x}(t)$ can be written as $\vec{x} = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t} + c_3\vec{v}_3 e^{\lambda_3 t} + \ldots + c_n\vec{v}_n e^{\lambda_n t}$ where λ_1, λ_2, λ_3, $\ldots$, λ_n and $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\ldots$, $\vec{v}_n$ are the eigenvalues and corresponding eigenvectors of the matrix A.

3. Phase Portraits With Straight Line Solutions

Exercise

Solve $\dfrac{dx}{dt} = 2x + 2y, \quad \dfrac{dy}{dt} = x + 3y.$

GroupWork

Use `HPGSystemSolver` (or `PPLANE`) to sketch the phase portrait of the linear system $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \vec{x}$ you solved above, in the space below.