Differential Equations

Math 341 Fall 2014 MWF 3:00-3:55pm Fowler 307
 `http://faculty.oxy.edu/ron/math/341/14/`

Worksheet 16

TITLE Linear Systems of ODEs
CURRENT READING Blanchard, 3.1

Homework Assignments due Friday October 17 (* indicates EXTRA CREDIT)
Section 3.1: 6, 7, 8, 10, 13, 18*.
Section 3.2: 8, 9, 12, 16*, 17, 18*.

SUMMARY
We will begin to analyze systems for ODEs which have the form $\frac{d\vec{x}}{dt} = A\vec{x}$.

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1. Constant Coefficient Linear Systems of ODEs

The **dimension** of a system of ODEs is equal to the number of independent variables in the system. Generally, in this chapter we shall be considering 2-dimensional systems of ODEs.

$$\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned}$$

This can be written as $\frac{d\vec{x}}{dt} = \frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A\vec{x}$ where $\vec{x} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.

Theorem

If A is a matrix with non-zero determinant, THEN the only equilibrium point for the linear system of ODEs $\frac{d\vec{x}}{dt} = A\vec{x}$ is the origin.

Exercise

Prove the above theorem.

2. Linearity Principles for Linear Systems of ODEs

There are two more linearity principles that apply to linear systems of ODEs of the form $\frac{d\vec{x}}{dt} = A\vec{x}$:

1. Given a solution $\vec{x}(t)$ of the system, then $k\vec{x}$ is also a solution.
2. Given two solutions $\vec{x}_1$ and $\vec{x}_2$ of the system, then $\vec{x}_1 + \vec{x}_2$ is also a solution.

EXAMPLE

Let's verify these two linearity principles.

GroupWork

(a) Show that $\frac{d\vec{x}}{dt} = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix} \vec{x}$ has $\vec{x}_1(t) = \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}$ and $\vec{x}_2(t) = \begin{bmatrix} -e^{-4t} \\ 2e^{-4t} \end{bmatrix}$ as solutions.

(b) Is $\vec{x}_3(t) = -2\vec{x}_1(t) + 5\vec{x}_2(t)$ also a solution?

(c) Which (if any) of these solutions solve the initial $\frac{d\vec{x}}{dt} = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix} \vec{x}$ with $\vec{x}(0) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$?

3. General Solution To IVP with a Linear System of ODEs

EXAMPLE

Let's find the general solution to $\frac{d\vec{x}}{dt} = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix} \vec{x}$ with $\vec{x}(0) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

Theorem: General Solution to Linear System of ODEs

Suppose $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are solutions of the linear equations $\frac{d\vec{x}}{dt} = A\vec{x}$. If $\vec{x}_1(0)$ and $\vec{x}_2(0)$ are linearly independent, then for any initial condition $\vec{x}_0$ we can find a general solution to the initial value problem $\frac{d\vec{x}}{dt} = A\vec{x}$ wth $\vec{x}(0) = \vec{x}_0$ which looks like $k_1\vec{x}_1(t) + k_2\vec{x}_2(t)$, a 2-parameter family of solutions, where k_1 and k_2 are arbitrary constants.

RECALL

A set of n vectors $\{\vec{v}_k\}$ are **linearly independent** IF AND ONLY IF the (only) solution to $\sum_{k=1}^{n} c_k\vec{v}_k = \vec{0}$ is $c_1 = c_2 = \ldots = c_n = 0$. In particular, the only way two (non-zero) vectors can be linearly independent if one of them is not a scalar multiple of the other.

EXAMPLE

Blanchard, page 262, # 27.

Consider the linear system $\frac{d\vec{Y}}{dt} = \begin{bmatrix} -2 & -1 \\ 2 & -5 \end{bmatrix} \vec{Y}$.

(a) Check that the functions $\vec{Y}_1(t) = \begin{bmatrix} e^{-3t} \\ e^{-3t} \end{bmatrix}$ and $\vec{Y}_2(t) = \begin{bmatrix} e^{-4t} \\ 2e^{-4t} \end{bmatrix}$ are solutions to the differential equation. (If they are not, stop!)

(c) Check that the two solutions are linearly independent. If they are not, stop!)

(c) Solve the initial value problem $\frac{d\vec{Y}}{dt} = \begin{bmatrix} -2 & -1 \\ 2 & -5 \end{bmatrix} \vec{Y}$, $\vec{Y}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.