Differential Equations

Math 341 Fall 2014 MWF 3:00-3:55pm Fowler 307

 http://faculty.oxy.edu/ron/math/341/14/

Worksheet 12

TITLE Analytic Solution Methods for Special Linear Systems
CURRENT READING Blanchard, 2.3

Homework Set #6 due Friday October 3 (* indicates EXTRA CREDIT)
Section 2.2: 7, 8, 11, 21* (EXPLAIN!), 24, 26. **Section 2.4**: 2, 5, 7, 8.
Section 2.5: 2, 3. **Chapter 2 Review**: 2, 3, 7, 12, 13 15, 16, 20, 30*.

SUMMARY
We will learn an analytical technique to obtain solutions of specific classes of linear systems (decoupled and partially decoupled).

1. Decoupling

A system of differential equations where one of the differential equations is actually autonomous (the rate of change of the dependent variable depends only on that dependent variable) the system is said to be **partially decoupled**. If *all* of the differential equations in the system are autonomous then the system is said to be **fully decoupled**.

Exercise Which of the folllowing systems are partially decoupled, decoupled or coupled? Label each one.

System A. $\dfrac{dx}{dt} = 2x + y; \;\; \dfrac{dy}{dt} = -y.$

System B. $\dfrac{dx}{dt} = 5x; \;\; \dfrac{dy}{dt} = -y.$

System C. $\dfrac{dx}{dt} = x + y; \;\; \dfrac{dy}{dt} = 4x - 2y.$

System D. $\dfrac{dx}{dt} = y; \;\; \dfrac{dy}{dt} = -\dfrac{k}{m}x.$

EXAMPLE

Blanchard, page 195, Question 7.

Again consider **System A** $\dfrac{dx}{dt} = 2x + y; \;\; \dfrac{dy}{dt} = -y$. Find the general solution.

Exercise

System A $\frac{dx}{dt} = 2x + y$; $\frac{dy}{dt} = -y$. Find the particular solution of System A that passes through the point $(1, 1)$.

2. Decoupled Systems Are Easier

Clearly, fully decoupled systems would be even simpler to solve.

EXAMPLE

Consider **System E** $\frac{dx}{dt} = 2x$ $\frac{dy}{dt} = -y$ with initial condition(s) $x(0) = 1, \quad y(0) = 2$.
Solve the system of IVPs.

3. Checking Solutions

Remember for a given function $\vec{x}(t)$ to satisfy a system of IVPs, it must satisfy **both** the initial condition $\vec{x}(t_0) = \vec{x_0}$ AND the ODEs $\frac{d\vec{x}}{dt} = \vec{F}(\vec{x})$.

Exercise

Blanchard, page 194, Question 6.

Is the function $\vec{Y}(t) = \begin{bmatrix} 4e^{2t} - e^{-t} \\ 3e^{-t} \end{bmatrix}$ a solution to System A ($x' = 2x + y;\ \ y' = -y$)?

4. Some Coupled Systems Are Easy To Solve

Consider the 2^{nd} order constant coefficient ODE $y''+py'+qy=0$ where p and q are constants. This ODE corresponds to the linear system

$$\frac{dy}{dt} = v; \;\; \frac{dv}{dt} = -pv - qy.$$

Although this is a fully coupled system, one can use a useful trick to solve the system because it is simpler to solve as a constant coefficient ODE.

For an n^{th}-order constant coefficient ODE you can make the guess that the solution looks like $y = e^{rt}$ and plug into the given equation.

EXAMPLE

Solve the equation $y'' + 5y' + 6y = 0$.

$$\begin{aligned} y'' + 5y' + 6y &= 0 \\ (e^{rt})'' + 5(e^{rt})' + 6e^{rt} &= 0 \\ r^2e^{rt} + 5re^{rt} + 6re^{rt} &= 0 \\ (r^2 + 5r + 6)e^{rt} &= 0 \\ (r+3)(r+2)e^{rt} &= 0 \end{aligned}$$

which means that $r = -3$ or $r = -2$.
The general solution to $y'' + 5y + 6y = 0$ is $y = Ae^{-2t} + Be^{-3t}$.
The polynomial $r^2 + 5r + 6 = 0$ is known as the characteristic polynomial of this 2nd order constant-coefficient ODE. Its roots tell you what exponents to use in the general solution.

NOTE: This "trick" only works with constant coefficient ODEs, and it will only work to solve systems of ODEs that can be written as such.

Exercise

Consider $y'' + y' + y = 0$. What 2-D system is it equivalent to? (HINT: what are values of p and q? What is it's characteristic polynomial?)

5. Connecting 2nd Order Linear ODEs and 2-D 1st Order Systems

DEFINITION: **The Wronskian**

Suppose the second order linear homogeneous differential equation $y'' + p(t)y' + q(t)y = 0$ has two solutions $x_1(t)$ and $x_2(t)$. We can define the **Wronskian** of x_1 and x_2, denoted $W(x_1, x_2, t)$ to be equal to $x_1(t)x_2'(t) - x_1'(t)x_2(t)$,

$$W(x_1, x_2) = \det \begin{bmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{bmatrix}$$

The Wronskian can be used to tell us that given two functions $x_1(t)$ and $x_2(t)$ that satisfy a linear homogeneous DE, the **general solution** to that DE will be $x(t) = C_1x_1(t) + C_2x_2(t)$.

`THEOREM`

If x_1 and x_2 are any two solutions of $x'' + p(t)x' + q(t)x = 0$ and their Wronskian $x_1(t)x_2'(t) - x_1'(t)x_2(t)$ is unequal to zero for all values of t, then $x(t) = C_1x_1(t) + C_2x_2(t)$ is the general solution to the DE. x_1 and x_2 are said to be a **fundamental solution set** of the given DE.

`THEOREM`

If x_1 and x_2 are any two solutions of $x'' + p(t)x' + q(t)x = 0$ then their Wronskian $x_1(t)x_2'(t) - x_1'(t)x_2(t)$ is an exponential function, and is either equal to zero for all r or unequal to zero for all t.

EXAMPLE

Recall from **Math 341 Fall 2014, Quiz 1** that the ODE

$$t^2x'' + 3tx' + x = 0$$

has proposed solutions $x_1 = \dfrac{1}{t}$ and $x_2(t) = \dfrac{\ln(t)}{t}$. Show that x_1 and x_2 are a fundamental set of solutions to the ODE.

EXTRA CREDIT HW

Show that the Wronskian $W(t) = x_1(t)x_2'(t) - x_1'(t)x_2(t)$ of the linear ODE $y'' + p(t)y' + q(t)y = 0$ itself satisfies the ODE $\frac{dW}{dt} = -p(t)W$ where $p(t)$ is the coefficient of the x' term!

This means that the Wronskian can be computed by the expression $W(t) = Ce^{-\int p(t)dt}$.

Confirm that the above formula can be used to compute the Wronskian for the ODE from Quiz 1,

$$t^2x'' + 3tx' + x = 0$$

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