Differential Equations

Math 341 Fall 2014 MWF 3:00-3:55pm Fowler 307

 http://faculty.oxy.edu/ron/math/341/14/

Class 5: Monday September 8

TITLE Existence and Uniqueness of Solutions

CURRENT READING Blanchard, 1.5

Homework Set #3 due Friday September 12

Section 1.4: 2, 6, 11, 15. Section 1.5: 2, 3, 12, 14, 15. Section 1.6: 2, 7, 8, 19, 20, 30, 31, 41.

SUMMARY

We will investigate the conditions which guarantee existence and/or uniqueness of solutions to the initial value problem $y' = f(t, y), \quad y(t_0) = y_0$.

1. Do Problems Always Have Solutions?

Think about the equation $2x^5 - 10x + 5 = 0$. Does it have a solution? How do we know?

DISCUSS

2. Existence and Uniqueness Of Particular Solutions

The main questions we would like to be able to answer when analyzing IVPs are:

1) **Existence** Does the differential equation possess solutions which pass through the given initial condition? and

2) **Uniqueness** If such a solution does exist, can we be certain that it is the only one? Luckily, there's a theorem that answers these questions for us.

THEOREM: **Existence of a Unique Solution**

Let $\mathcal{R}$ be a rectangular region in the xy-plane defined by $a \le x \le b, c \le y \le d$ that contains the point (x_0, y_0) in its interior. IF $f(x, y)$ and $\partial f/\partial y$ are continuous on $\mathcal{R}$, THEN there exists some interval I_0 defined as $x_0 - h < x < x_0 + h$ for $h > 0$ contained in $a \le x \le b$ and a unique function $y(x)$ defined on I_0 that is a solution of the initial value problem $y' = f(x, y) \quad y(x_0) = y_0$.

3. Theorems Have Hypotheses and Conclusions

The existence and uniqueness theorem is actually two different theorems with different hypotheses and conclusions.

1) **Existence** IF $f(t, y)$ is continuous on a square containing (t_0, y_0) THEN **there exists** a solution on an interval $(t_0 - \epsilon, t_0 + \epsilon)$ for some $\epsilon > 0$

2) **Uniqueness** IF $f(t, y)$ and $\dfrac{\partial f}{\partial y}$ are both continuous on a square containing (t_0, y_0) THEN there exists **a unique solution** on an interval $(t_0 - \epsilon, t_0 + \epsilon)$ for some $\epsilon > 0$

RECALL

IF A, THEN B **is equivalent to** (The Contrapositive) IF NOT B, THEN NOT A.

(The Inverse) IF NOT A, THEN NOT B and (The Converse) IF B, THEN A are equivalent to each other, but are NOT equivalent to the original theorem $A \Rightarrow B$.

For example consider the logical statement: "IF it is raining, THEN the ground is wet."

The contrapositive is "If the ground is NOT wet, then it is NOT raining."

The Converse: IF ______________________ THEN ________________.

The Inverse: IF ____________________ THEN ____________________________.

EXAMPLE

Show that the initial value problem

$$\frac{dy}{dx} = x\sqrt{y}, \quad y(0) = 0$$

has at least two solutions since the trivial solution $y(x) = 0$ and the solution $y(x) = \frac{1}{16}x^4$ both satisfy the IVP. **Verify this!**

Using the Existence and Uniqueness Theorem, we look at the functions $f(x, y) = x\sqrt{y}$ and $\dfrac{\partial f}{\partial y} = \dfrac{x}{2\sqrt{y}}$. At the origin $(0, 0)$ what can we say about $f(x, y)$ and $f_y(x, y)$?

What can we say about $f(x, y)$ and $f_y(x, y)$ at $(2, 4)$? What does this imply about existence and uniqueness of the corresponding IVP $y' = xy^{1/2}, y(2) = 4$?

4. Implications of the Existence & Uniqueness Theorem (EUT)

LEMMA: **When Solution Curves Do Not Intersect**

IF $y' = f(x, y)$ is a first-order differential equation with f and $\frac{\partial f}{\partial x}$ both continuous for **all** values of x and y in some region $\mathcal{S}$ in the xy-plane,THEN inside the region $\mathcal{S}$ the solution curves of the differential equation will form a non-intersecting space-filling family of curves.

Exercise

Inspired by **Blanchard, Devaney & Hall, #9, page 72**.

(a) Show that $y_1(t) = t^2$ and $y_2(t) = t^2+1$ are both solutions of $\dfrac{dy}{dt} = -y^2+y+2yt^2+2t-t^2-t^4$

(b) Show that if $y(t)$ is another solution to the given ODE with initial condition $0 < y(0) < 1$ then $t^2 < y(t) < t^2 + 1$ for all t

(c) Illustrate your answer by using technology to explore the slope field