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# Differential Equations

Math 341 Fall 2010  
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MWF 2:30-3:25pm Fowler 307  
<http://faculty.oxy.edu/ron/math/341/10/>

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## Worksheet 24: Wednesday November 10

**TITLE** Dissipative Systems

**CURRENT READING** Blanchard, 5.3 & 5.4

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### Homework Assignments due Friday November 12

Section 5.1: 3, 4, 5, 18, 21.

Section 5.3: 2, 12, 13, 14, 17, 18.

Chapter 5 Review: 3, 4, 5, 6, 7, 8, 11, 12, 25, 27, 28.

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### SUMMARY

We shall continue our analysis of non-linear systems by introducing the concept of a Lyapunov function and learn about gradient systems.

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### EXAMPLE

Recall the ODE for the damped harmonic oscillator  $y'' + py' + qy = 0$  written as a system of ODEs

$$\begin{aligned}\frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -pv - qy\end{aligned}$$

Recall that the function  $H(y, v) = \frac{1}{2}v^2 + \frac{1}{2}qy^2$  is a Hamiltonian for the system when  $p = 0$ .

However, what is  $\frac{dH}{dt}$  now?

$$\begin{aligned}\frac{dH}{dt} &= \frac{\partial H}{\partial y} \frac{dy}{dt} + \frac{\partial H}{\partial v} \frac{dv}{dt} \\ &= (qy)v + v(-pv - qy) \\ &= -pv^2\end{aligned}$$

Which, when  $p > 0$  implies that the quantity  $H(y, v) = \frac{1}{2}v^2 + \frac{1}{2}qy^2$  decreases with time along solution curves of the given system. Such a function is not known as a Hamiltonian function but a Lyapunov function. Lyapunov functions are often used to make conclusions about the stability of equilibria of nonlinear systems of DEs.

### DEFINITION: Lyapunov Function

A function  $L(x, y)$  is called a **Lyapunov function** for a system of differential equations, if, for every solution  $(x(t), y(t))$  that is not an equilibrium solution of the system,

$$\frac{d}{dt}L(x(t), y(t)) \leq 0$$

for every  $t$  with strict inequality except for a discrete set of values for  $t$ .

## 1. Gradient Systems

A system of differential equations is known as a **gradient system** if there exists a function  $G(x, y)$  such that for every  $(x, y)$

$$\begin{aligned}\frac{dx}{dt} &= \frac{\partial G}{\partial x} \\ \frac{dy}{dt} &= \frac{\partial G}{\partial y}\end{aligned}$$

If  $(x(t), y(t))$  are solutions of gradient system, then

$$\begin{aligned}\frac{dG}{dt} &= \frac{\partial G}{\partial x} \frac{dx}{dt} + \frac{\partial G}{\partial y} \frac{dy}{dt} \\ &= \frac{\partial G}{\partial x} \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} \frac{\partial G}{\partial y} \\ &= (G_x)^2 + (G_y)^2 \\ &\geq 0\end{aligned}$$

This should makes us realize that if we want to form a Lyapunov function for a gradient system all we need to do is select  $L(x, y) = -G(x, y)$ ! So, **all gradient systems possess a Lyapunov function.** ( The converse is NOT true, i.e. every system with a Lyapunov function is NOT a gradient system.)

### **EXAMPLE**

Let's show that  $L(x, y) = -G(x, y)$  is a Lyapunov function for any gradient system  $\dot{x} = G_x, \quad \dot{y} = G_y$ .

**Exercise****Blanchard, Devaney & Hall, 5.4.1, page 518.**

Consider

$$\begin{aligned}\frac{dx}{dt} &= -x^3 \\ \frac{dy}{dt} &= -y^3\end{aligned}$$

(a) Show that  $L(x, y) = \frac{1}{2}(x^2 + y^2)$  is a Lyapunov function for the given system.

(b) Sketch the level sets of  $L(x, y)$

(c) What can you conclude about the phase portrait of the system given your information from (a) and (b)?

## 2. Properties of Gradient Systems

### Gradient Systems can not possess periodic solutions!

By using Linearization, we can show that the eigenvalues of the Jacobian of a gradient system evaluated at its equilibria will always be real (i.e. not complex) and thus solution curves of gradient systems will never be periodic.

### Not All Systems That Have Lyapunov Functions Are Gradient Systems

#### **EXAMPLE**

The following system has a Lyapunov function of  $L(x, y) = x^2 + y^2$  but is NOT a gradient system.

$$\begin{aligned}\frac{dx}{dt} &= -x + y \\ \frac{dy}{dt} &= -x - y\end{aligned}$$

Let's Prove This Result.