Differential Equations

Math 341 Fall 2010 MWF 2:30-3:25pm Fowler 307
 `http://faculty.oxy.edu/ron/math/341/10/`

Worksheet 12: **Friday October 1**

TITLE Analytic Solution Methods for Special Linear Systems
CURRENT READING Blanchard, 2.3

Homework Assignments due Friday October 8
Section 2.2: 7, 8, 10, 24, 25.
Section 2.3: 3, 4, 7, 11.
Section 2.4: 2, 4.
Chapter 2 Review: 2, 10, 11, 13, 14, 19, 20, 28, 30.

SUMMARY
We will learn an analytical technique to obtain solutions of specific classes of linear systems (decoupled and partially decoupled).

1. Decoupling

A system of differential equations where one of the differential equations is actually autonomous (the rate of change of the dependent variable depends only on that dependent variable) the system is said to be **partially decoupled**. If *all* of the differential equations in the system are autonomous then the system is said to be **fully decoupled**.

Exercise Which of the folllowing systems are partially decoupled, decoupled or coupled? Label each one.

System A. $\frac{dx}{dt} = 2x + y;\ \ \frac{dy}{dt} = -y.$

System B. $\frac{dx}{dt} = 5x;\ \ \frac{dy}{dt} = -y.$

System C. $\frac{dx}{dt} = x + y;\ \ \frac{dy}{dt} = 4x - 2y.$

System D. $\frac{dx}{dt} = y;\ \ \frac{dy}{dt} = -\frac{k}{m}x.$

EXAMPLE
Blanchard, page 196, Question 7.
Again consider **System A** $\frac{dx}{dt} = 2x + y;\ \ \frac{dy}{dt} = -y$. Find the general solution.

Exercise

System A $\dfrac{dx}{dt} = 2x + y$; $\dfrac{dy}{dt} = -y$. Find the particular solution of System A that passes through the point $(1, 1)$.

2. Decoupled Systems Are Easier

Clearly, completely decoupled systems would be even simpler to solve.

EXAMPLE

Consider **System E** $\dfrac{dx}{dt} = 2x$ $\dfrac{dy}{dt} = -y$ with initial condition $x(0) = 1, \quad y(0) = 2.$
Solve the system.

3. Checking Solutions

Remember for a given function $\vec{x}(t)$ to satisfy a system of ODEs, it must satisfy **both** the initial condition $\vec{x}(t_0) = \vec{x_0}$ AND the ODEs $\dfrac{d\vec{x}}{dt} = \vec{F}(\vec{x})$.

Exercise

Blanchard, page 196, Question 6.

Is the function $\vec{Y}(t) = \begin{bmatrix} 4e^{2t} - e^{-t} \\ 3e^{-t} \end{bmatrix}$ a solution to System A?

4. Some Coupled Systems Are Easy

Consider the 2^{nd} order constant coefficient ODE $y''+py'+qy=0$ where p and q are constants. This ODE corresponds to the linear system

$$\frac{dy}{dt}=v;\ \ \frac{dv}{dt}=-pv-qy.$$

Although this is a fully coupled system, one can use a useful trick to solve the system because it is simpler to solve as a constant coefficient ODE.

For an n^{th}-order constant coefficient ODE you can make the guess that the solution looks like $y=e^{rt}$ and plug into the given equation.

EXAMPLE

Solve the equation $y''+5y'+6y=0$.

$$\begin{aligned} y''+5y'+6y &= 0 \\ (e^{rt})''+5(e^{rt})'+6e^{rt} &= 0 \\ r^2e^{rt}+5re^{rt}+6re^{rt} &= 0 \\ (r^2+5r+6)e^{rt} &= 0 \\ (r+3)(r+2)e^{rt} &= 0 \end{aligned}$$

which means that $r=-3$ or $r=-2$.
The general solution to $y''+5y+6y=0$ is $y=Ae^{-2t}+Be^{-3t}$.

NOTE: This "trick" only works with constant coefficient ODEs, and it will only work to solve systems of ODEs that can be written as such.

Exercise

Solve $y''+y'+y=0$.