Math 341 Fall 2008 **SHOW ALL YOUR WORK** Quiz SEVEN

1. The **Gamma Function** is defined as

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1}\, dt, \quad (\alpha > 0).$$

(a) *1 point.* Show that $\Gamma(1) = 1$.

$$\Gamma(1) = \int_0^\infty e^{-t} t^{1-1} dt = \lim_{b\to\infty} \int_0^b e^{-t} dt = \lim_{b\to\infty} -e^{-t}\Big|_0^b$$
$$= \lim_{b\to\infty} -e^{-b} + 1 = 1$$

(b) *2 points.* Show that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

$$\Gamma(\alpha+1) = \int_0^\infty e^{-t} t^{\alpha} dt = t^\alpha e^{-t}\Big|_0^\infty - \int_0^\infty (-e^{-t})\alpha t^{\alpha-1} dt$$

$u = t^\alpha$ $du = \alpha t^{\alpha-1}$
$dv = e^{-t}$ $v = -e^{-t}$

$$= \lim_{b\to\infty} b^\alpha e^{-b} - 0 + \alpha\int_0^\infty e^{-t} t^{\alpha-1} dt$$
$$= 0 + \alpha\Gamma(\alpha)$$
$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

(c) *3 points.* Use your results in (a) and (b) to show that $\Gamma(n+1) = n!$, where n is a positive integer. (HINT: use mathematical induction).

Induction: $\Gamma(2) = 2\Gamma(1) = 2\cdot 1 = 2!$ ($n=1$ case)

$\Gamma(n) = (n-1)!$ $\Rightarrow \Gamma(n+1) = n!$

$\Gamma(n) = (n-1)\Gamma(n-1)$

$\Gamma(n+1) = n\,\Gamma(n) \Rightarrow \Gamma(n+1) = n\cdot(n-1)\Gamma(n-1) = n(n-1)(n-2)! \cdots \Gamma(2)$

$= n\cdot(n-1)(n-2) \cdots$

$= n(n-1)(n-2)\cdots 2\cdot 1$

$= n!$

(d) *4 points.* Use all the previous results to show that $\mathcal{L}[t^\alpha] = \dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$ and $\mathcal{L}[t^n] = \dfrac{n!}{s^{n+1}}$ when n is a positive integer.

$$\mathcal{L}[t^\alpha] = \int_0^\infty e^{-st} t^\alpha dt = \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^\alpha \frac{du}{s} = \frac{1}{s^{\alpha+1}}\int_0^\infty e^{-u} u^\alpha du = \frac{\Gamma(\alpha+1)}{s^\alpha}$$

let $u = st$ $du = s\,dt$

$$\mathcal{L}[t^n] = \int_0^\infty e^{-st} t^n dt = t^n \frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} n t^{n-1} dt = \frac{n}{s}\int_0^\infty t^{n-1} e^{-st} dt$$

$u = t^n$ $du = nt^{n-1}$
$dv = e^{-st}$ $v = \frac{e^{-st}}{-s}$

$\mathcal{L}[t^n] = \frac{n}{s}\mathcal{L}[t^{n-1}]$

$\mathcal{L}[t^0] = \mathcal{L}[1] = \frac{1}{s}$

$\mathcal{L}[t^n] = \frac{n!}{s^n}\cdot\frac{1}{s} = \frac{n!}{s^{n+1}}$