Math 341 Fall 2010 **SHOW ALL YOUR WORK** Quiz Five

1. Consider the linear system of ordinary differential equations with a real-valued parameter a

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 2 & a \\ 2 & 0 \end{bmatrix} \vec{x} \text{ where } \vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

Let's describe how the phase portrait changes as a varies from $-\infty$ to $+\infty$.

(a) *3 points.* Compute all the eigenvalues and eigenvectors in order to sketch the phase portrait when $a = -3/2$. Describe the stationary point at the origin ~~when $a = a_B$~~.

$\lambda^2 - 2\lambda - 2a = 0 \Rightarrow \lambda = \dfrac{2 \pm \sqrt{(-2)^2 - 4(1)(-2a)}}{2} = \dfrac{2 \pm \sqrt{4+8a}}{2} = 1 \pm \sqrt{1+2a}$

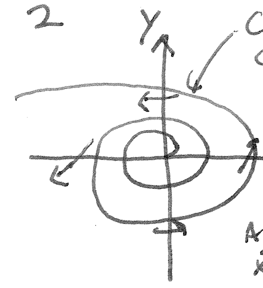
$T = 2$
$D = -2a$

$a = -3/2$.
$\lambda = 1 \pm \sqrt{1-3}$
$= 1 \pm \sqrt{-2}$
$= 1 \pm i\sqrt{2}$
$\lambda = 1 + i\sqrt{2},\ 1 - i\sqrt{2}$

NO REAL EIGENVECTORS

when $x=1, y=1$ $\dot{x} = 1/2, \dot{y} = 2$ $\Rightarrow \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = 1$

UNSTABLE SPIRAL SOURCE at ORIGIN



At $x = 0$
$\dot{x} = -3/2\, y$
$\dot{y} = 0$

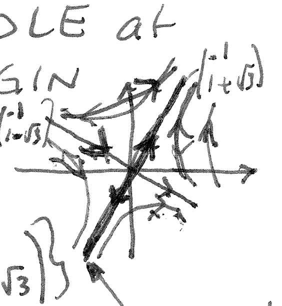
(b) *3 points.* Compute all the eigenvalues and eigenvectors in order to sketch the phase portrait when $a = 1$. Describe the stationary point at the origin.

$a = 1,\ \lambda = 1 \pm \sqrt{1+2} = 1 \pm \sqrt{3} \Rightarrow \lambda = 1+\sqrt{3},\ 1-\sqrt{3}$ SADDLE at ORIGIN

$\lambda = 1+\sqrt{3}$
$A - \lambda I \Rightarrow \left(\begin{array}{cc|c} 1-\sqrt{3} & 1 & 0 \\ 2 & -1-\sqrt{3} & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1-\sqrt{3} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow E_\lambda = \text{span}\left\{\begin{pmatrix} -1 \\ 1-\sqrt{3} \end{pmatrix}\right\}$

$\lambda = 1-\sqrt{3}$
$A - \lambda I \Rightarrow \left(\begin{array}{cc|c} 1+\sqrt{3} & 1 & 0 \\ 2 & -1+\sqrt{3} & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1+\sqrt{3} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow E_{1-\sqrt{3}} = \text{span}\left\{\begin{pmatrix} -1 \\ 1+\sqrt{3} \end{pmatrix}\right\}$

$\vec{x}(t) = c_1 e^{(1+\sqrt{3})t}\begin{pmatrix} -1 \\ 1-\sqrt{3} \end{pmatrix} + c_2 e^{(1-\sqrt{3})t}\begin{pmatrix} -1 \\ 1+\sqrt{3} \end{pmatrix}$



UNSTABLE

dominant direction

(c) *4 points.* For what value of a does the system change its nature (i.e. bifurcate)? Call this value a_B and compute the eigenvalues and eigenvectors in order to sketch the phase portrait for $a = a_B$. Describe the stationary point at the origin when $a = a_B$.

when $1 + 2a = 0 \Rightarrow a = -1/2$ the system bifurcates

$a = a_B = -\frac{1}{2} \Rightarrow \lambda = 1$

$(A - \lambda I)\vec{x} = \vec{0}$ $\left(\begin{array}{cc|c} 1 & -1/2 & 0 \\ 2 & -1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow 2x_1 - x_2 = 0$, $2x_1 = x_2$, x_1 free

$\vec{x} = \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}, x_1 \in \mathbb{R}$

The origin in this case is an improper node

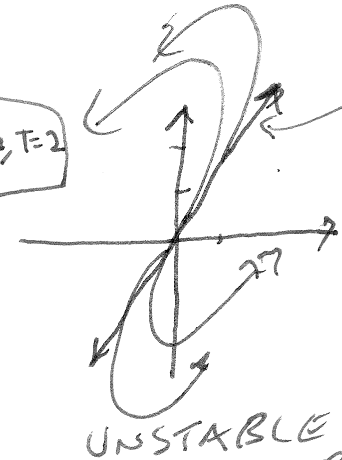
Find generalized eigenvector

$(A - \lambda I)\vec{w} = \vec{v}$

$\begin{pmatrix} 1 & -1/2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow 2w_1 - w_2 = 2$
$w_1 = 1$
$w_2 = 0$

$\vec{x}(t) = c_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^t \left[t\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$

NOTE: Since $D = -2a$, $T = 2$ another bifurcation occurs at $a = 0$



dominant direction $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

UNSTABLE IMPROPER NODE