Math 341 Fall 2010 **SHOW ALL YOUR WORK** Quiz **Two**

1. Consider the following one-parameter family of nonlinear, first-order, autonomous differential equations where α is a known real parameter value

$$\frac{dy}{dx} = y^2 - \alpha y + 1.$$

(a) *2 points.* Show that this DE has no equilibrium values when $|\alpha| < 2$.

$y^2 - \alpha y + 1 = 0 \Rightarrow y^* = \dfrac{\alpha \pm \sqrt{\alpha^2 - 4}}{2}$

$\alpha^2 - 4 < 0$

$\alpha^2 < 4$

$\pm\alpha < 2 \Rightarrow -2 < \alpha < 2$ or $|\alpha| < 2$

Clearly, when $\alpha^2 - 4 < 0$ no real equilibrium values will exist

(b) *2 points.* For what values of α will the DE have exactly one equilibrium value? Classify the equilibrium point (as node, source or sink) in this case and write down the constant solution.

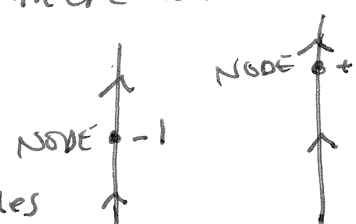
When $\alpha^2 - 4 = 0$ there will be one equilibrium value, $y = \frac{\alpha}{2}$.

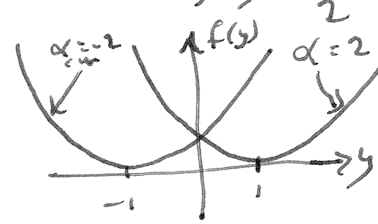
$\alpha^2 = 4$

$\alpha = \pm 2$

$y^* = \pm 1$

Both will be nodes since $f(y) > 0$ everywhere else





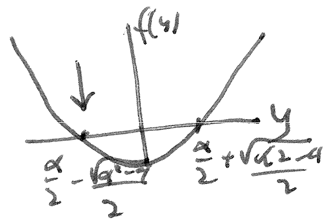
(c) *4 points.* Show that when $|\alpha| > 2$ the DE has exactly one stable equilibrium value (sink) and one unstable equilibrium value (source). Show work that supports your classification of the equilibria, and sketch a phase line for a representative value of α.

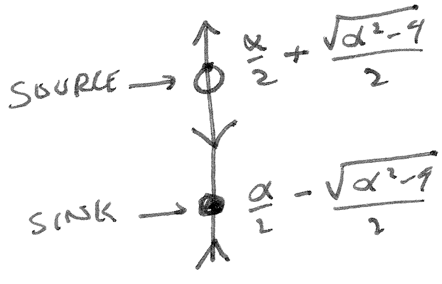
When $|\alpha| > 2$

$y^* = \dfrac{\alpha}{2} + \dfrac{\sqrt{\alpha^2 - 4}}{2}$

and

$y^* = \dfrac{\alpha}{2} - \dfrac{\sqrt{\alpha^2 - 4}}{2}$





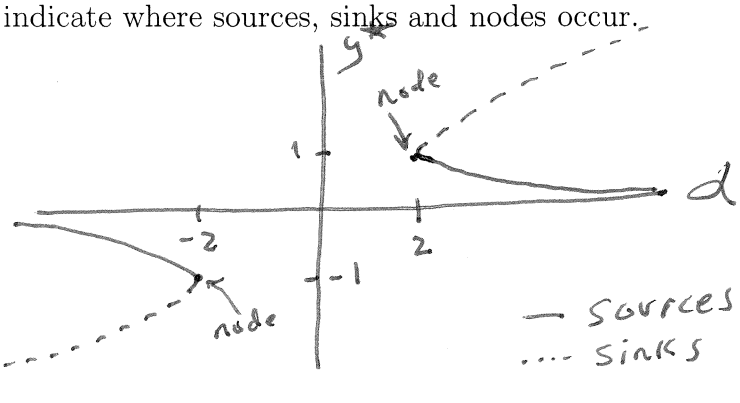
$f(y) = -\alpha y + y^2 + 1$

$f'(y) = -\alpha + 2y$

$f'(y^* = \frac{\alpha}{2} +) = -\frac{\alpha}{2} + \sqrt{\alpha^2 - 4} > 0 \Rightarrow$ (UNSTABLE SOURCE

$f'(y^* = \frac{\alpha}{2} -) = -\frac{\alpha}{2} - \sqrt{\alpha^2 - 4} < 0 \Rightarrow$ SINK (STABLE

(d) *2 points.* Use your answers from above to sketch the bifurcation diagram for the given DE. Clearly indicate where sources, sinks and nodes occur.



$y^* = \dfrac{\alpha}{2} \pm \sqrt{\left(\dfrac{\alpha}{2}\right)^2 - 1}$

Note: $\lim_{\alpha \to \infty} y^* = 0$

$\lim_{\alpha \to -\infty} y^* = 0$