6.4

35. **(a)** Consider

$$\mathcal{L}[f] = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt.$$

We can calculate dF/ds by differentiating under the integral sign. That is,

$$\begin{aligned}\frac{dF}{ds} &= \int_0^\infty \frac{\partial}{\partial s}\left(f(t)\, e^{-st}\right) dt \\ &= \int_0^\infty f(t)(-t)e^{-st}\, dt \\ &= -\mathcal{L}[tf(t)].\end{aligned}$$

(b) If we apply this result to

$$\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2} = \omega(s^2 + \omega^2)^{-1},$$

we obtain

$$\begin{aligned}\mathcal{L}[t \sin \omega t] &= -\omega(-1)(s^2 + \omega^2)^{-2}(2s) \\ &= \frac{2\omega s}{(s^2 + \omega^2)^2}.\end{aligned}$$

Compare this result with the result of Exercise 6.

XERCISES FOR SECTION 6.4

1. This is the $\frac{0}{0}$ case of L'Hôpital's Rule. Differentiating numerator and denominator with respect to Δt, we obtain

$$\frac{se^{s\Delta t} - (-s)e^{-s\Delta t}}{2},$$

which simplifies to

$$\frac{s(e^{s\Delta t} + e^{-s\Delta t})}{2}.$$

Since both $e^{s\Delta t}$ and $e^{-s\Delta t}$ tend to 1 as $\Delta t \to 0$, the desired limit is s.

2. Taking Laplace transforms of both sides and applying the rules yields

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 3\mathcal{L}[y] = 5\mathcal{L}[\delta_2].$$

Simplifying, using the initial conditions, and the fact that $\mathcal{L}[\delta_2] = e^{-2s}$, we get

$$(s^2 + 3)\mathcal{L}[y] = 5e^{-2s}.$$

Hence,

$$\mathcal{L}[y] = 5\frac{e^{-2s}}{s^2+3}.$$

This can be written as

$$\mathcal{L}[y] = \frac{5}{\sqrt{3}}e^{-2s}\frac{\sqrt{3}}{s^2+3},$$

which yields

$$y(t) = \frac{5}{\sqrt{3}}u_2(t)\sin\left(\sqrt{3}(t-2)\right).$$

3. Applying the Laplace transform to both sides, using the rules, and the fact that $\mathcal{L}[\delta_3] = e^{-3s}$, we get

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 2s\mathcal{L}[y] - 2y(0) + 5\mathcal{L}[y] = e^{-3s}.$$

Substituting the given initial conditions, we have

$$\mathcal{L}[y] = \frac{s+3}{s^2+2s+5} + \frac{e^{-3s}}{s^2+2s+5}.$$

Using the fact that $s^2+2s+5 = (s+1)^2+4$, we obtain

$$\mathcal{L}[y] = \frac{s+1}{(s+1)^2+4} + \frac{2}{(s+1)^2+4} + \frac{1}{2}e^{-3s}\frac{2}{(s+1)^2+4}.$$

Therefore,

$$y(t) = e^{-t}\cos 2t + e^{-t}\sin 2t + \frac{1}{2}u_3(t)e^{-(t-3)}\sin(2(t-3)).$$

4. Taking the Laplace transform of both sides, using the rules, and the fact that $\mathcal{L}[\delta_2] = e^{-2s}$, we get

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 2s\mathcal{L}[y] - 2y(0) + 2\mathcal{L}[y] = -2e^{-2s}.$$

Substituting the given initial conditions, we obtain

$$\mathcal{L}[y] = \frac{2s+4}{s^2+2s+2} - \frac{2e^{-2s}}{s^2+2s+2}.$$

Using $s^2+2s+2 = (s+1)^2+1$ in the denominator gives us

$$\mathcal{L}[y] = 2\frac{s+1}{(s+1)^2+1} + 2\frac{1}{(s+1)^2+1} - 2e^{-2s}\frac{1}{(s+1)^2+1}.$$

Taking the inverse Laplace transform, we have

$$y(t) = 2e^{-t}\cos t + 2e^{-t}\sin t - 2u_2(t)e^{-(t-2)}\sin(t-2).$$

5. Applying Laplace transform to both sides, using the rules, and the fact that $\mathcal{L}[\delta_a] = e^{-as}$, we get

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 2s\mathcal{L}[y] - 2y(0) + 3\mathcal{L}[y] = e^{-s} - 3e^{-4s}.$$

Substituting the initial conditions gives us

$$\mathcal{L}[y] = \frac{e^{-s}}{s^2+2s+3} - \frac{3e^{-4s}}{s^2+2s+3}.$$

Now, using that $s^2 + 2s + 3 = (s+1)^2 + 2$, we have

$$\mathcal{L}[y] = \frac{1}{\sqrt{2}}e^{-s}\frac{\sqrt{2}}{(s+1)^2+2} + \frac{3}{\sqrt{2}}e^{-4s}\frac{\sqrt{2}}{(s+1)^2+2}.$$

So,

$$y(t) = \frac{1}{\sqrt{2}}u_1(t)e^{-(t-1)}\sin(\sqrt{2}(t-1)) - \frac{3}{\sqrt{2}}u_4(t)e^{-(t-4)}\sin(\sqrt{2}(t-4)).$$

6. **(a)** The characteristic polynomial of the unforced oscillator is $\lambda^2 + 2\lambda + 3$, and the eigenvalues are $\lambda = -1 \pm \sqrt{2}\,i$. Hence, the natural period is $\sqrt{2}\,\pi$ and the damping causes the solutions of the unforced equation to tend to zero like e^{-t}. At $t = 4$, the system is given a jolt, so the solution rises. After $t = 4$, the equation is unforced, so the solution again tends to zero as e^{-t}.

(b) Taking Laplace transforms of both sides of the equation, we have

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 2s\mathcal{L}[y] - 2y(0) + 3\mathcal{L}[y] = \mathcal{L}[\delta_4].$$

Plugging in the initial conditions and solving for $\mathcal{L}[y]$ gives us

$$\mathcal{L}[y] = \frac{s+2}{s^2+2s+3} + \frac{e^{-4s}}{s^2+2s+3}.$$

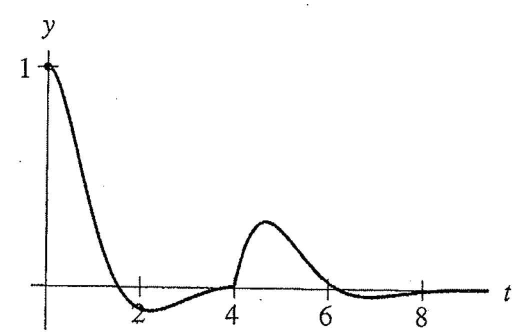
If we complete the square for the polynomial s^2+2s+3, we get $s^2+2s+3 = (s+1)^2+2$, so

$$\mathcal{L}[y] = \frac{s+1}{(s+1)^2+2} + \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+2} + \frac{1}{\sqrt{2}}e^{-4s}\frac{\sqrt{2}}{(s+1)^2+2}.$$

Therefore,

$$y(t) = e^{-t}\cos\sqrt{2}\,t + \frac{1}{\sqrt{2}}e^{-t}\sin\sqrt{2}\,t + \frac{1}{\sqrt{2}}u_4(t)e^{-(t-4)}\sin(\sqrt{2}(t-4)).$$

(c)



Note that the solution goes through about 3/4 of a natural period before the application of the delta function. The delta function forcing causes the second maximum of the solution to be much higher than it would have been without the forcing, but the long term effect is small because the damping is fairly large.

7. **(a)** From the table

$$\mathcal{L}[\delta_a] = e^{-as}$$

$$s\mathcal{L}[u_a] - u_a(0) = s\frac{e^{-as}}{s} - 0 = e^{-as}.$$

(b) The formula for the Laplace transform of a derivative is

$$\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$$

and this is exactly the relationship between the Laplace transforms of $u_a(t)$ and $\delta_a(t)$. Hence, it is tempting to think of the Dirac delta function as the derivative of the Heaviside function.

(c) We can think of the Heaviside function $u_a(t)$ as a limit of piecewise linear functions equal to zero for t less than $a - \Delta t$, equal to one for t greater than $a + \Delta t$ and a straight line for t between $a - \Delta t$ and $a + \Delta t$. The derivative of this function is precisely the function $g_{\Delta t}$ used to define the Dirac delta function. This is still just an informal relationship until we specify in what sense we are taking the limit.

8. Actually, this exercise is a little more complicated than it seems at first. We can think of g as a periodic function with period a and apply Exercise 16 in Section 6.2, but to do so, we must decide how to integrate $\delta_a(t)$ over the interval $0 \leq t \leq a$. In other words, is the impulse inside or outside the interval?

To avoid this issue, we consider the function

$$f(t) = \sum_{n=0}^{\infty} \delta_{na+a/2}(t).$$

We can apply the periodicity formula from Exercise 16 in Section 6.2 to this function to get

$$\mathcal{L}[f] = \frac{1}{1 - e^{-as}} \int_0^a f(t)\, e^{-st}\, dt = \frac{1}{1 - e^{-as}} \int_0^a \delta_{a/2}(t)\, e^{-st}\, dt,$$

because $\delta_{na+a/2}(t) = 0$ for all $n > 0$ on the interval $[0, a]$. Moreover,

$$\int_0^a \delta_{a/2}(t)\, e^{-st}\, dt = \mathcal{L}[\delta_{a/2}]$$

because $\delta_{a/2}(t) = 0$ for all $t > a/2$. Therefore, we have

$$\mathcal{L}[f] = \frac{e^{-as/2}}{1 - e^{-as}}.$$

To obtain $\mathcal{L}[g]$, we use the relation $g(t) = u_{a/2}(t) f(t - a/2)$ to obtain

$$\mathcal{L}[g] = e^{-as/2}\frac{e^{-as/2}}{1 - e^{-as}} = \frac{e^{-as}}{1 - e^{-as}}.$$

Note that this is the same answer we get if we apply the periodicity formula directly to $g(t)$ assuming that the entire impulse takes place inside the interval $0 \leq t \leq a$. In other words, if we assume that

$$\int_0^a \delta_a(t)\, e^{-st} dt = e^{-as},$$

then we get

$$\begin{aligned}\mathcal{L}[g] &= \frac{1}{1-e^{-as}} \int_0^a g(t)\, e^{-st}\, dt \\ &= \frac{1}{1-e^{-as}} \int_0^a \delta_a(t)\, e^{-st}\, dt \\ &= \frac{e^{-as}}{1-e^{-as}}.\end{aligned}$$

9. **(a)** To compute the Laplace transform of the infinite sum on the right-hand side of the equation, we can either sum the geometric series that results from the fact that $\mathcal{L}[\delta_n] = e^{-ns}$ or use Exercise 16 in Section 6.2. Either way, we get

$$\mathcal{L}\left[\sum_{n=1}^{\infty} \delta_n(t)\right] = \frac{e^{-s}}{1-e^{-s}} = \frac{1}{e^s - 1}.$$

For our purposes, it is actually better to leave the Laplace transform of the right-hand side as

$$\mathcal{L}\left[\sum_{n=1}^{\infty} \delta_n(t)\right] = \sum_{n=1}^{\infty} e^{-ns}.$$

Since $y(0) = 0$ and $y'(0) = 0$, the transformed equation is

$$s^2\mathcal{L}[y] + 2\mathcal{L}[y] = \sum_{n=1}^{\infty} e^{-ns},$$

which simplifies to

$$\mathcal{L}[y] = \frac{1}{s^2+2} \sum_{n=1}^{\infty} e^{-ns} = \sum_{n=1}^{\infty} \frac{e^{-ns}}{s^2+2}.$$

(b) Since

$$\mathcal{L}^{-1}\left[\frac{e^{-ns}}{s^2+2}\right] = \frac{1}{\sqrt{2}} u_n(t) \sin(\sqrt{2}\,(t-n)),$$

we have

$$y(t) = \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} u_n(t) \sin(\sqrt{2}\,(t-n)).$$

(c) The period of the forcing is different from the natural period of the unforced oscillator. Hence, the solution oscillates but not periodically.

10. **(a)** To compute the Laplace transform of the infinite sum on the right-hand side of the equation, we can either sum the geometric series or use Exercise 16 in Section 6.2 (see Exercise 9 as well). We get

$$\mathcal{L}\left[\sum_{n=1}^{\infty} \delta_{2n\pi}(t)\right] = \frac{e^{-2\pi}}{1-e^{-2\pi}}.$$