Then substituting back we have

$$\mathcal{L}[e^{at}\cos\omega t] = \frac{s-a}{(s-a)^2+\omega^2}.$$

5. Using the formula

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] = s^2\mathcal{L}[y] - y'(0) - sy(0),$$

and the linearity of the Laplace transform, we get that

$$s^2\mathcal{L}[y] - y'(0) - sy(0) + \omega^2\mathcal{L}[y] = 0.$$

Substituting the initial conditions and solving for $\mathcal{L}[y]$ gives

$$\mathcal{L}[y] = \frac{s}{s^2+\omega^2}.$$

6. Since

$$\mathcal{L}[\cos\omega t] = \frac{s}{s^2+\omega^2},$$

we can compute that

$$\frac{d}{d\omega}\mathcal{L}[\cos\omega t] = \frac{-s(2\omega)}{(s^2+\omega^2)^2} = \frac{-2\omega s}{(s^2+\omega^2)^2},$$

but

$$\frac{d}{d\omega}\mathcal{L}[\cos\omega t] = \mathcal{L}\left[\frac{d}{d\omega}\cos\omega t\right] = \mathcal{L}[-t\sin\omega t].$$

We can bring the derivative with respect to ω inside the Laplace transform because the Laplace transform is an integral with respect to t, that is,

$$\frac{d}{d\omega}\mathcal{L}[\cos\omega t] = \frac{d}{d\omega}\int_0^\infty \cos\omega t\, e^{-st}\,dt = \int_0^\infty \frac{d}{d\omega}\left(\cos\omega t\, e^{-st}\right)dt.$$

Canceling the minus signs on left and right gives

$$\mathcal{L}[t\sin\omega t] = \frac{2\omega s}{(s^2+\omega^2)^2}.$$

7. Since

$$\mathcal{L}[\sin\omega t] = \frac{\omega}{s^2+\omega^2},$$

we can compute that

$$\frac{d}{d\omega}\mathcal{L}[\sin\omega t] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2},$$

but

$$\frac{d}{d\omega}\mathcal{L}[\sin\omega t] = \mathcal{L}\left[\frac{d}{d\omega}\sin\omega t\right] = \mathcal{L}[t\cos\omega t].$$

So

$$\mathcal{L}[t\cos\omega t] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}.$$

8. We need to compute

$$\mathcal{L}[te^{at}] = \int_0^\infty te^{at}e^{-st}\,dt.$$

We can do this using the hint, by differentiating $\mathcal{L}[e^{at}]$ with respect to a. Another method is to write

$$\mathcal{L}[te^{at}] = \int_0^\infty te^{at}e^{-st}\,dt = \int_0^\infty te^{-(s-a)t}\,dt = \int_0^\infty te^{-rt}\,dt$$

where $r = s - a$. The last integral is the Laplace transform of t using r as the new independent variable. Hence, from the table we have

$$\int_0^\infty te^{-rt}\,dt = \frac{1}{r^2}.$$

Substituting back $r = s - a$ we have

$$\mathcal{L}[te^{at}] = \frac{1}{(s-a)^2}.$$

9. From Exercise 10, we know that

$$\mathcal{L}[te^{at}] = \frac{1}{(s-a)^2}.$$

Differentiating both sides of this formula with respect to a gives

$$\frac{d}{da}\mathcal{L}[te^{at}] = \mathcal{L}\left[\frac{d}{da}te^{ta}\right] = \mathcal{L}[t^2e^{at}]$$

while

$$\frac{d}{da}\frac{1}{(s-a)^2} = \frac{2}{(s-a)^3}.$$

Hence,

$$\mathcal{L}[t^2e^{at}] = \frac{2}{(s-a)^3}.$$

10. Using the results of Exercise 9, we can work by induction on n, with induction hypothesis

$$\mathcal{L}[t^ne^{at}] = \frac{n!}{(s-a)^{n+1}}.$$

Alternately, we can compute

$$\mathcal{L}[t^ne^{at}] = \int_0^\infty t^ne^{at}e^{-st}\,dt = \int_0^\infty t^ne^{-(s-a)t}\,dt = \int_0^\infty t^ne^{-rt}\,dt$$

where $r = s - a$. Now the last integral is the Laplace transform of t^n using r as the independent variable, so

$$\int_0^\infty t^ne^{-rt}\,dt = \frac{n!}{r^{n+1}}$$

from the table. Hence, substituting $r = s - a$ we have

$$\mathcal{L}[t^ne^{at}] = \frac{n!}{(s-a)^{n+1}}.$$

11. In this case, $b = 2$, and $(s + b/2)^2 = (s+1)^2 = s^2 + 2s + 1$, so $s^2 + 2s + 10 = (s+1)^2 + 3^2$.

12. In this case, $b = -4$, and $(s + b/2)^2 = (s-2)^2 = s^2 - 4s + 4$, so $s^2 - 4s + 5 = (s-2)^2 + 1^2$.

13. In this case, $b = 1$, and $(s + b/2)^2 = (s+1/2)^2 = s^2 + s + 1/4$, so $s^2 + s + 1 = (s+1/2)^2 + 3/4 = (s+1/2)^2 + (\sqrt{3}/2)^2$.

14. In this case, $b = 6$, and $(s + b/2)^2 = (s+3)^2 = s^2 + 6s + 9$, so $s^2 + 6s + 10 = (s+3)^2 + 1^2$.

15. In Exercise 11, we completed the square and obtained $s^2 + 2s + 10 = (s+1)^2 + 3^2$, so

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{s^2+2s+10}\right] &= \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+3^2}\right] \\ &= \frac{1}{3}\mathcal{L}^{-1}\left[\frac{3}{(s+1)^2+3^2}\right] \\ &= \frac{1}{3}e^{-t}\sin 3t.\end{aligned}$$

16. In Exercise 12, we completed the square and obtained $s^2 - 4s + 5 = (s-2)^2 + 1^2$, so

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{s}{s^2-4s+5}\right] &= \mathcal{L}^{-1}\left[\frac{s}{(s-2)^2+1^2}\right] \\ &= \mathcal{L}^{-1}\left[\frac{s-2}{(s-2)^2+1^2}\right] + \mathcal{L}^{-1}\left[\frac{2}{(s-2)^2+1^2}\right] \\ &= e^{2t}\cos t + e^{2t}(2\sin t) = e^{2t}(\cos t + 2\sin t).\end{aligned}$$

17. In Exercise 13, we completed the square and obtained $s^2 + s + 1 = (s+1/2)^2 + (\sqrt{3}/2)^2$, so

$$\frac{2s+3}{s^2+s+1} = \frac{2s+3}{(s+1/2)^2+(\sqrt{3}/2)^2}.$$

We want to put this fraction in the right form so that we can use the formulas for $\mathcal{L}[e^{at}\cos\omega t]$ and $\mathcal{L}[e^{at}\sin\omega t]$. We see that

$$\begin{aligned}\frac{2s+3}{(s+1/2)^2+(\sqrt{3}/2)^2} &= \frac{2s+1}{(s+1/2)^2+(\sqrt{3}/2)^2} + \frac{2}{(s+1/2)^2+(\sqrt{3}/2)^2} \\ &= \frac{2(s+1/2)}{(s+1/2)^2+(\sqrt{3}/2)^2} + \frac{(4/\sqrt{3})(\sqrt{3}/2)}{(s+1/2)^2+(\sqrt{3}/2)^2}.\end{aligned}$$

So

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{2s+3}{s^2+s+1}\right] &= 2\mathcal{L}^{-1}\left[\frac{(s+1/2)}{(s+1/2)^2+(\sqrt{3}/2)^2}\right] + \frac{4}{\sqrt{3}}\mathcal{L}^{-1}\left[\frac{\sqrt{3}/2}{(s+1/2)^2+(\sqrt{3}/2)^2}\right] \\ &= 2e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{4}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right).\end{aligned}$$

). In Exercise 14, we completed the square and obtained $s^2 + 6s + 10 = (s+3)^2 + 1^2$, so

$$\frac{s+1}{s^2+6s+10} = \frac{s+1}{(s+3)^2+1^2}.$$

We want to put this fraction in the right form so that we can use the formulas for $\mathcal{L}[e^{at}\cos\omega t]$ and $\mathcal{L}[e^{at}\sin\omega t]$. We see that

$$\frac{s+1}{(s+3)^2+1^2} = \frac{s+3}{(s+3)^2+1^2} - \frac{2}{(s+3)^2+1^2}.$$

So

$$\mathcal{L}^{-1}\left[\frac{s+1}{s^2+6s+10}\right] = e^{-3t}\cos t - 2e^{-3t}\sin t.$$

). We compute

$$\begin{aligned}\mathcal{L}\left[e^{(a+ib)t}\right] &= \int_0^\infty e^{(a+ib)t}e^{-st}\,dt\\ &= \int_0^\infty e^{-(s-(a+ib))t}\,dt\\ &= -\frac{1}{s-(a+ib)}\left(\lim_{u\to\infty}\left[e^{-(s-a)u}e^{-ibu}\right]-1\right).\end{aligned}$$

The limit is zero as long as $s > a$. Hence,

$$\mathcal{L}\left[e^{(a+ib)t}\right] = \frac{1}{s-(a+ib)}$$

if $s > a$ and undefined otherwise. This is the same formula as for real exponentials. It can also be written

$$\mathcal{L}\left[e^{(a+ib)t}\right] = \frac{s-a+ib}{(s-a)^2+b^2}.$$

0. This follows from linearity:

$$\begin{aligned}\mathcal{L}[y] &= \mathcal{L}[y_{\mathrm{re}} + iy_{\mathrm{im}}]\\ &= \int_0^\infty (y_{\mathrm{re}} + iy_{\mathrm{im}})\,e^{-st}\,dt\\ &= \int_0^\infty y_{\mathrm{re}}(t)\,e^{-st}\,dt + i\int_0^\infty y_{\mathrm{im}}(t)\,e^{-st}\,dt\\ &= \mathcal{L}[y_{\mathrm{re}}] + i\mathcal{L}[y_{\mathrm{im}}].\end{aligned}$$

Taking inverse Laplace transforms of the right-hand side gives

$$\left(1-\tfrac{2}{\sqrt{3}}i\right)e^{(-1+i\sqrt{3})t/2}+\left(1+\tfrac{2}{\sqrt{3}}i\right)e^{(-1-i\sqrt{3})t/2}.$$

Using Euler's formula to replace the complex exponentials and simplifying yields

$$2e^{-t/2}\cos\left(\tfrac{\sqrt{3}}{2}t\right)+\tfrac{4}{\sqrt{3}}e^{-t/2}\sin\left(\tfrac{\sqrt{3}}{2}t\right).$$

26. Using the quadratic formula, we find the roots of the denominator are $-3\pm i$ so the denominator can be factored

$$s^2+6s+10=(s-(-3+i))(s-(-3-i)).$$

The partial fractions decomposition is

$$\frac{s+1}{s^2+6s+10}=\frac{A}{s-(-3-i)}+\frac{B}{s-(-3+i)},$$

which leads to the equations

$$\begin{cases} A+B=1 \\ (3-i)A+(3+i)B=1. \end{cases}$$

Solving, we find $A=\frac{1}{2}-i$ and $B=\frac{1}{2}+i$, so

$$\frac{s+1}{s^2+6s+10}=\frac{\frac{1}{2}-i}{s-(-3-i)}+\frac{\frac{1}{2}+i}{s-(-3+i)}.$$

Taking inverse Laplace transform of the right-hand side gives

$$\left(\tfrac{1}{2}-i\right)e^{(-3-i)t}+\left(\tfrac{1}{2}+i\right)e^{(-3+i)t}$$

and using Euler's formula and simplifying gives

$$e^{-3t}\cos t-2e^{-3t}\sin t.$$

27. **(a)** Taking the Laplace transform of both sides of the equation, we obtain

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right]+4\mathcal{L}[y]=\frac{8}{s},$$

and using the fact that $\mathcal{L}[d^2y/dt^2]=s^2\mathcal{L}[y]-sy(0)-y'(0)$, we have

$$(s^2+4)\mathcal{L}[y]-sy(0)-y'(0)=\frac{8}{s}.$$

(b) Substituting the initial conditions yields

$$(s^2+4)\mathcal{L}[y] - 11s - 5 = \frac{8}{s},$$

and solving for $\mathcal{L}[y]$ we get

$$\mathcal{L}[y] = \frac{11s+5}{s^2+4} + \frac{8}{s(s^2+4)}.$$

The partial fractions decomposition of $8/(s(s^2+4))$ is

$$\frac{8}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}.$$

Putting the right-hand side over a common denominator gives us

$$(A+B)s^2 + Cs + 4A = 8,$$

and consequently, $A = 2$, $B = -2$, and $C = 0$. In other words,

$$\frac{8}{s(s^2+4)} = \frac{2}{s} + \frac{-2s}{s^2+4}.$$

We obtain

$$\mathcal{L}[y] = \frac{2}{s} + \frac{9s+5}{s^2+4}.$$

(c) To take the inverse Laplace transform, we rewrite $\mathcal{L}[y]$ in the form

$$\mathcal{L}[y] = \frac{2}{s} + 9\left(\frac{s}{s^2+4}\right) + \frac{5}{2}\left(\frac{2}{s^2+4}\right).$$

Therefore, $y(t) = 2 + 9\cos 2t + \frac{5}{2}\sin 2t$.

28. **(a)** Taking the Laplace transform of both sides of the equation, we obtain

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] - \mathcal{L}[y] = \frac{1}{s-2},$$

and using the fact that $\mathcal{L}[d^2y/dt^2] = s^2\mathcal{L}[y] - sy(0) - y'(0)$, we have

$$(s^2-1)\mathcal{L}[y] - sy(0) - y'(0) = \frac{1}{s-2}.$$

(b) Substituting the initial conditions yields

$$(s^2-1)\mathcal{L}[y] - s + 1 = \frac{1}{s-2},$$

and solving for $\mathcal{L}[y]$ we get

$$\mathcal{L}[y] = \frac{1}{s+1} + \frac{1}{(s-2)(s^2-1)}.$$

Using the partial fractions decomposition

$$\frac{1}{(s-2)(s^2-1)} = \frac{\frac{1}{3}}{s-2} + \frac{-\frac{1}{2}}{s-1} + \frac{\frac{1}{6}}{s+1},$$

we obtain

$$\mathcal{L}[y] = \frac{\frac{1}{3}}{s-2} + \frac{-\frac{1}{2}}{s-1} + \frac{\frac{7}{6}}{s+1}.$$

(c) Taking the inverse Laplace transform, we have

$$y(t) = \tfrac{1}{3}e^{2t} - \tfrac{1}{2}e^{t} + \tfrac{7}{6}e^{-t}.$$

29. **(a)** Taking the Laplace transform of both sides of the equation, we obtain

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] - 4\mathcal{L}\left[\frac{dy}{dt}\right] + 5\mathcal{L}[y] = \frac{2}{s-1},$$

and using the formulas for $\mathcal{L}[dy/dt]$ and $\mathcal{L}[d^2y/dt^2]$ in terms of $\mathcal{L}[y]$, we have

$$(s^2-4s+5)\mathcal{L}[y] - sy(0) - y'(0) + 4y(0) = \frac{2}{s-1}.$$

(b) Substituting the initial conditions yields

$$(s^2-4s+5)\mathcal{L}[y] - 3s + 11 = \frac{2}{s-2},$$

and solving for $\mathcal{L}[y]$ we get

$$\mathcal{L}[y] = \frac{3s-11}{s^2-4s+5} + \frac{2}{(s-1)(s^2-4s+5)}.$$

Using the partial fractions decomposition

$$\frac{2}{(s-1)(s^2-4s+5)} = \frac{1}{s-1} + \frac{-s+3}{s^2-4s+5},$$

we obtain

$$\mathcal{L}[y] = \frac{1}{s-1} + \frac{2s-8}{s^2-4s+5}.$$

(c) In order to compute the inverse Laplace transform, we first write

$$s^2-4s+5 = (s-2)^2+1$$

by completing the square, and then we write

$$\frac{2s-8}{s^2-4s+5} = \frac{2(s-2)}{(s-2)^2+1} - \frac{4}{(s-2)^2+1}.$$

Taking the inverse Laplace transform, we have

$$y(t) = e^t + 2e^{2t}\cos t - 4e^{2t}\sin t.$$

35. **(a)** Consider

$$\mathcal{L}[f] = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt.$$

We can calculate dF/ds by differentiating under the integral sign. That is,

$$\begin{aligned}\frac{dF}{ds} &= \int_0^\infty \frac{\partial}{\partial s}\left(f(t)\, e^{-st}\right) dt \\ &= \int_0^\infty f(t)(-t)e^{-st}\, dt \\ &= -\mathcal{L}[tf(t)].\end{aligned}$$

(b) If we apply this result to

$$\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2+\omega^2} = \omega(s^2+\omega^2)^{-1},$$

we obtain

$$\begin{aligned}\mathcal{L}[t \sin \omega t] &= -\omega(-1)(s^2+\omega^2)^{-2}(2s) \\ &= \frac{2\omega s}{(s^2+\omega^2)^2}.\end{aligned}$$

Compare this result with the result of Exercise 6.

EXERCISES FOR SECTION 6.4

1. This is the $\frac{0}{0}$ case of L'Hôpital's Rule. Differentiating numerator and denominator with respect to Δt, we obtain

$$\frac{se^{s\Delta t} - (-s)e^{-s\Delta t}}{2},$$

which simplifies to

$$\frac{s(e^{s\Delta t} + e^{-s\Delta t})}{2}.$$

Since both $e^{s\Delta t}$ and $e^{-s\Delta t}$ tend to 1 as $\Delta t \to 0$, the desired limit is s.

2. Taking Laplace transforms of both sides and applying the rules yields

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 3\mathcal{L}[y] = 5\mathcal{L}[\delta_2].$$

Simplifying, using the initial conditions, and the fact that $\mathcal{L}[\delta_2] = e^{-2s}$, we get

$$(s^2+3)\mathcal{L}[y] = 5e^{-2s}.$$

Hence,

$$\mathcal{L}[y] = 5\frac{e^{-2s}}{s^2+3}.$$

This can be written as

$$\mathcal{L}[y] = \frac{5}{\sqrt{3}}e^{-2s}\frac{\sqrt{3}}{s^2+3},$$

which yields

$$y(t) = \frac{5}{\sqrt{3}}u_2(t)\sin\left(\sqrt{3}(t-2)\right).$$

3. Applying the Laplace transform to both sides, using the rules, and the fact that $\mathcal{L}[\delta_3] = e^{-3s}$, we get

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 2s\mathcal{L}[y] - 2y(0) + 5\mathcal{L}[y] = e^{-3s}.$$

Substituting the given initial conditions, we have

$$\mathcal{L}[y] = \frac{s+3}{s^2+2s+5} + \frac{e^{-3s}}{s^2+2s+5}.$$

Using the fact that $s^2+2s+5 = (s+1)^2+4$, we obtain

$$\mathcal{L}[y] = \frac{s+1}{(s+1)^2+4} + \frac{2}{(s+1)^2+4} + \frac{1}{2}e^{-3s}\frac{2}{(s+1)^2+4}.$$

Therefore,

$$y(t) = e^{-t}\cos 2t + e^{-t}\sin 2t + \frac{1}{2}u_3(t)e^{-(t-3)}\sin(2(t-3)).$$

4. Taking the Laplace transform of both sides, using the rules, and the fact that $\mathcal{L}[\delta_2] = e^{-2s}$, we get

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 2s\mathcal{L}[y] - 2y(0) + 2\mathcal{L}[y] = -2e^{-2s}.$$

Substituting the given initial conditions, we obtain

$$\mathcal{L}[y] = \frac{2s+4}{s^2+2s+2} - \frac{2e^{-2s}}{s^2+2s+2}.$$

Using $s^2+2s+2 = (s+1)^2+1$ in the denominator gives us

$$\mathcal{L}[y] = 2\frac{s+1}{(s+1)^2+1} + 2\frac{1}{(s+1)^2+1} - 2e^{-2s}\frac{1}{(s+1)^2+1}.$$

Taking the inverse Laplace transform, we have

$$y(t) = 2e^{-t}\cos t + 2e^{-t}\sin t - 2u_2(t)e^{-(t-2)}\sin(t-2).$$

5. Applying Laplace transform to both sides, using the rules, and the fact that $\mathcal{L}[\delta_a] = e^{-as}$, we get

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 2s\mathcal{L}[y] - 2y(0) + 3\mathcal{L}[y] = e^{-s} - 3e^{-4s}.$$

Substituting the initial conditions gives us

$$\mathcal{L}[y] = \frac{e^{-s}}{s^2+2s+3} - \frac{3e^{-4s}}{s^2+2s+3}.$$

Now, using that $s^2 + 2s + 3 = (s+1)^2 + 2$, we have

$$\mathcal{L}[y] = \frac{1}{\sqrt{2}} e^{-s} \frac{\sqrt{2}}{(s+1)^2+2} + \frac{3}{\sqrt{2}} e^{-4s} \frac{\sqrt{2}}{(s+1)^2+2}.$$

So,

$$y(t) = \frac{1}{\sqrt{2}} u_1(t) e^{-(t-1)} \sin(\sqrt{2}(t-1)) - \frac{3}{\sqrt{2}} u_4(t) e^{-(t-4)} \sin(\sqrt{2}(t-4)).$$

6. **(a)** The characteristic polynomial of the unforced oscillator is $\lambda^2 + 2\lambda + 3$, and the eigenvalues are $\lambda = -1 \pm \sqrt{2}\,i$. Hence, the natural period is $\sqrt{2}\pi$ and the damping causes the solutions of the unforced equation to tend to zero like e^{-t}. At $t = 4$, the system is given a jolt, so the solution rises. After $t = 4$, the equation is unforced, so the solution again tends to zero as e^{-t}.

(b) Taking Laplace transforms of both sides of the equation, we have

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 2s\mathcal{L}[y] - 2y(0) + 3\mathcal{L}[y] = \mathcal{L}[\delta_4].$$

Plugging in the initial conditions and solving for $\mathcal{L}[y]$ gives us

$$\mathcal{L}[y] = \frac{s+2}{s^2+2s+3} + \frac{e^{-4s}}{s^2+2s+3}.$$

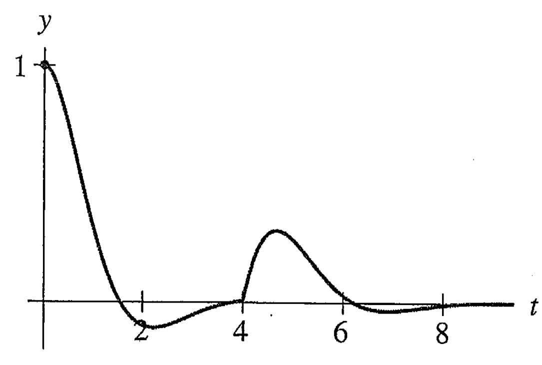
If we complete the square for the polynomial s^2+2s+3, we get $s^2+2s+3 = (s+1)^2+2$, so

$$\mathcal{L}[y] = \frac{s+1}{(s+1)^2+2} + \frac{1}{\sqrt{2}} \frac{\sqrt{2}}{(s+1)^2+2} + \frac{1}{\sqrt{2}} e^{-4s} \frac{\sqrt{2}}{(s+1)^2+2}.$$

Therefore,

$$y(t) = e^{-t}\cos\sqrt{2}t + \frac{1}{\sqrt{2}} e^{-t} \sin\sqrt{2}t + \frac{1}{\sqrt{2}} u_4(t) e^{-(t-4)} \sin(\sqrt{2}(t-4)).$$

(c)



Note that the solution goes through about 3/4 of a natural period before the application of the delta function. The delta function forcing causes the second maximum of the solution to be much higher than it would have been without the forcing, but the long term effect is small because the damping is fairly large.

7. **(a)** From the table

$$\mathcal{L}[\delta_a] = e^{-as}$$

$$s\mathcal{L}[u_a] - u_a(0) = s\frac{e^{-as}}{s} - 0 = e^{-as}.$$

(b) The formula for the Laplace transform of a derivative is

$$\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$$

and this is exactly the relationship between the Laplace transforms of $u_a(t)$ and $\delta_a(t)$. Hence, it is tempting to think of the Dirac delta function as the derivative of the Heaviside function.

(c) We can think of the Heaviside function $u_a(t)$ as a limit of piecewise linear functions equal to zero for t less than $a - \Delta t$, equal to one for t greater than $a + \Delta t$ and a straight line for t between $a - \Delta t$ and $a + \Delta t$. The derivative of this function is precisely the function $g_{\Delta t}$ used to define the Dirac delta function. This is still just an informal relationship until we specify in what sense we are taking the limit.

8. Actually, this exercise is a little more complicated than it seems at first. We can think of g as a periodic function with period a and apply Exercise 16 in Section 6.2, but to do so, we must decide how to integrate $\delta_a(t)$ over the interval $0 \le t \le a$. In other words, is the impulse inside or outside the interval?

To avoid this issue, we consider the function

$$f(t) = \sum_{n=0}^{\infty} \delta_{na+a/2}(t).$$

We can apply the periodicity formula from Exercise 16 in Section 6.2 to this function to get

$$\mathcal{L}[f] = \frac{1}{1-e^{-as}} \int_0^a f(t)\, e^{-st}\, dt = \frac{1}{1-e^{-as}} \int_0^a \delta_{a/2}(t)\, e^{-st}\, dt,$$

because $\delta_{na+a/2}(t) = 0$ for all $n > 0$ on the interval $[0, a]$. Moreover,

$$\int_0^a \delta_{a/2}(t)\, e^{-st}\, dt = \mathcal{L}[\delta_{a/2}]$$

because $\delta_{a/2}(t) = 0$ for all $t > a/2$. Therefore, we have

$$\mathcal{L}[f] = \frac{e^{-as/2}}{1-e^{-as}}.$$

To obtain $\mathcal{L}[g]$, we use the relation $g(t) = u_{a/2}(t) f(t - a/2)$ to obtain

$$\mathcal{L}[g] = e^{-as/2}\frac{e^{-as/2}}{1-e^{-as}} = \frac{e^{-as}}{1-e^{-as}}.$$

Note that this is the same answer we get if we apply the periodicity formula directly to $g(t)$ assuming that the entire impulse takes place inside the interval $0 \le t \le a$. In other words, if we assume that

$$\int_0^a \delta_a(t)\, e^{-st} dt = e^{-as},$$

then we get

$$\begin{aligned}\mathcal{L}[g] &= \frac{1}{1-e^{-as}} \int_0^a g(t)\, e^{-st}\, dt \\ &= \frac{1}{1-e^{-as}} \int_0^a \delta_a(t)\, e^{-st}\, dt \\ &= \frac{e^{-as}}{1-e^{-as}}.\end{aligned}$$

9. **(a)** To compute the Laplace transform of the infinite sum on the right-hand side of the equation, we can either sum the geometric series that results from the fact that $\mathcal{L}[\delta_n] = e^{-ns}$ or use Exercise 16 in Section 6.2. Either way, we get

$$\mathcal{L}\left[\sum_{n=1}^{\infty} \delta_n(t)\right] = \frac{e^{-s}}{1-e^{-s}} = \frac{1}{e^s - 1}.$$

For our purposes, it is actually better to leave the Laplace transform of the right-hand side as

$$\mathcal{L}\left[\sum_{n=1}^{\infty} \delta_n(t)\right] = \sum_{n=1}^{\infty} e^{-ns}.$$

Since $y(0) = 0$ and $y'(0) = 0$, the transformed equation is

$$s^2\mathcal{L}[y] + 2\mathcal{L}[y] = \sum_{n=1}^{\infty} e^{-ns},$$

which simplifies to

$$\mathcal{L}[y] = \frac{1}{s^2+2} \sum_{n=1}^{\infty} e^{-ns} = \sum_{n=1}^{\infty} \frac{e^{-ns}}{s^2+2}.$$

(b) Since

$$\mathcal{L}^{-1}\left[\frac{e^{-ns}}{s^2+2}\right] = \frac{1}{\sqrt{2}} u_n(t) \sin(\sqrt{2}\,(t-n)),$$

we have

$$y(t) = \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} u_n(t) \sin(\sqrt{2}\,(t-n)).$$

(c) The period of the forcing is different from the natural period of the unforced oscillator. Hence, the solution oscillates but not periodically.

10. **(a)** To compute the Laplace transform of the infinite sum on the right-hand side of the equation, we can either sum the geometric series or use Exercise 16 in Section 6.2 (see Exercise 9 as well). We get

$$\mathcal{L}\left[\sum_{n=1}^{\infty} \delta_{2n\pi}(t)\right] = \frac{e^{-2\pi}}{1-e^{-2\pi}}.$$