EXERCISES FOR SECTION 6.1

1. We have

$$\begin{aligned}
\mathcal{L}[3] &= \int_0^\infty 3e^{-st}\,dt \\
&= \lim_{b\to\infty} \int_0^b 3e^{-st}\,dt \\
&= \lim_{b\to\infty} \left(\frac{-3}{s} e^{-st} \Big|_0^b \right) \\
&= \lim_{b\to\infty} -\frac{3}{s} \left(e^{-sb} - e^0 \right) \\
&= \frac{3}{s} \quad \text{if } s > 0,
\end{aligned}$$

since $\lim_{b\to\infty} e^{-sb} = \lim_{b\to\infty} 1/e^{sb} = 0$ if $s > 0$.

2. We have

$$\mathcal{L}[t] = \int_0^\infty te^{-st}\,dt = \lim_{b\to\infty} \int_0^b te^{-st}\,dt.$$

To evaluate the integral we use integration by parts with $u = t$ and $dv = e^{-st}dt$. Then $du = dt$ and $v = -e^{-st}/s$. Thus

$$\begin{aligned}
\lim_{b\to\infty} \int_0^b te^{-st}\,dt &= \lim_{b\to\infty} \left(-\frac{te^{-st}}{s}\Big|_0^b - \int_0^b -\frac{e^{-st}}{s}\,dt \right) \\
&= \lim_{b\to\infty} \left(-\frac{be^{-sb}}{s} - \frac{e^{-st}}{s^2}\Big|_0^b \right) \\
&= \lim_{b\to\infty} \left(-\frac{be^{-sb}}{s} - \frac{e^{-sb}}{s^2} + \frac{e^0}{s^2} \right) \\
&= \frac{1}{s^2}
\end{aligned}$$

since

$$\lim_{b\to\infty} -\frac{be^{-sb}}{s} = \lim_{b\to\infty} \frac{-b}{se^{sb}} = \lim_{b\to\infty} \frac{-1}{s^2e^{sb}} = 0$$

by L'Hôpital's Rule if $s > 0$.

3. We use the fact that $\mathcal{L}[df/dt] = s\mathcal{L}[f] - f(0)$. Letting $f(t) = t^2$ we have $f(0) = 0$ and

$$\mathcal{L}[2t] = s\mathcal{L}[t^2] - 0$$

or

$$2\mathcal{L}[t] = s\mathcal{L}[t^2]$$

using the fact that the Laplace transform is linear. Then since $\mathcal{L}[t] = 1/s^2$ (by the previous exercise), we have

$$\mathcal{L}[-5t^2] = -5\mathcal{L}[t^2] = -5\left(\frac{2\mathcal{L}[t]}{s}\right) = -\frac{10}{s^3}.$$

4. We have shown thus far that $\mathcal{L}[t] = 1/s^2$ and $\mathcal{L}[t^2] = 2/s^3$. Let's compute $\mathcal{L}[t^3]$ and see if a pattern emerges. Using $\mathcal{L}[df/dt] = s\mathcal{L}[f] - f(0)$ with $f(t) = t^3$, we have

$$\mathcal{L}[3t^2] = 3\mathcal{L}[t^2] = s\mathcal{L}[t^3] - f(0)$$

which yields

$$\mathcal{L}[t^3] = \frac{3}{s}\mathcal{L}[t^2] = \frac{3 \cdot 2}{s^4} = \frac{3!}{s^4}.$$

If we were to continue, we would see that

$$\mathcal{L}[t^4] = \frac{4}{s}\mathcal{L}[t^3] = \frac{4!}{s^5}.$$

A clear pattern has emerged (which we could prove by induction should we be so inclined):

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

so that $\mathcal{L}[t^5] = 5!/s^6$.

5. To show a rule by induction, we need two steps. First, we need to show the rule is true for $n = 1$. Then, we need to show that if the rule holds for n, then it holds for $n+1$.

(a) $n = 1$. We need to show that $\mathcal{L}[t] = 1/s^2$.

$$\mathcal{L}[t] = \int_0^\infty te^{-st}\,dt.$$

Using integration by parts with $u = t$ and $dv = e^{-st}\,dt$, we find

$$\begin{aligned}
\mathcal{L}[t] &= \left.\frac{te^{-st}}{-s}\right|_0^\infty + \int_0^\infty \frac{e^{-st}}{s}\,dt \\
&= \lim_{b\to\infty}\left[\left.\frac{te^{-st}}{-s}\right|_0^b\right] + \int_0^\infty \frac{e^{-st}}{s}\,dt \\
&= \int_0^\infty \frac{e^{-st}}{s}\,dt \\
&= -\left.\frac{e^{-st}}{s^2}\right|_0^\infty \\
&= \frac{1}{s^2} \quad (s > 0).
\end{aligned}$$

(b) Now we assume that the rule holds for n, that is, that $\mathcal{L}[t^n] = n!/s^{n+1}$, and show it holds true for $n+1$, that is, $\mathcal{L}[t^{n+1}] = (n+1)!/s^{n+2}$. There are two different methods to do so:

(i)

$$\mathcal{L}[t^{n+1}] = \int_0^\infty t^{n+1}e^{-st}\,dt$$

Using integration by parts with $u = t^{n+1}$ and $dv = e^{-st}\,dt$, we find

$$\mathcal{L}[t^{n+1}] = -\frac{t^{n+1}e^{-st}}{s}\bigg|_0^\infty + \int_0^\infty \frac{n+1}{s} t^n\, e^{-st}\,dt.$$

Now,

$$\begin{aligned}
-\frac{t^{n+1}e^{-st}}{s}\bigg|_0^\infty &= \lim_{b\to\infty}\left[-\frac{t^{n+1}e^{-st}}{s}\bigg|_0^b\right] \\
&= \lim_{b\to\infty}\frac{-b^{n+1}e^{-sb}}{s} + 0 \\
&= 0 \quad (s > 0).
\end{aligned}$$

So

$$\begin{aligned}
\mathcal{L}[t^{n+1}] &= \int_0^\infty \frac{n+1}{s} t^n\, e^{-st}\,dt \\
&= \frac{n+1}{s}\int_0^\infty t^n\, e^{-st}\,dt \\
&= \frac{n+1}{s}\mathcal{L}[t^n].
\end{aligned}$$

Since we assumed that $\mathcal{L}[t^n] = n!/s^{n+1}$, we get that

$$\mathcal{L}[t^{n+1}] = \frac{n+1}{s}\cdot\frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}}$$

which is what we wanted to show.

(ii) We use the fact that $\mathcal{L}[df/dt] = s\mathcal{L}[f] - f(0)$. Letting $f(t) = t^{n+1}$ we have $f(0) = 0$ and

$$\mathcal{L}[(n+1)t^n] = s\mathcal{L}[t^{n+1}] - 0$$

or

$$(n+1)\mathcal{L}[t^n] = s\mathcal{L}[t^{n+1}]$$

using the fact that the Laplace transform is linear. Since we assumed $\mathcal{L}[t^n] = n!/s^{n+1}$, we have

$$\mathcal{L}[t^{n+1}] = \frac{n+1}{s}\mathcal{L}[t^n] = \frac{n+1}{s}\cdot\frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}},$$

which is what we wanted to show.

6. Using the formula that $\mathcal{L}[t^n] = n!/s^{n+1}$, we can see by linearity that

$$\mathcal{L}[a_i t^i] = a_i\frac{i!}{s^{i+1}},$$

so

$$\begin{aligned}
\mathcal{L}[a_0 + a_1 t + \cdots + a_n t^n] &= \mathcal{L}[a_0] + \mathcal{L}[a_1 t] + \cdots + \mathcal{L}[a_n t^n] \\
&= \frac{a_0}{s} + \frac{a_1}{s^2} + \cdots + \frac{a_n n!}{s^{n+1}}.
\end{aligned}$$

7. Since we know that $\mathcal{L}[e^{at}] = 1/(s-a)$, we have $\mathcal{L}[e^{3t}] = 1/(s-3)$, and therefore,

$$\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] = e^{3t}.$$

8. We see that

$$\frac{5}{3s} = \frac{5}{3}\cdot\frac{1}{s},$$

so

$$\mathcal{L}^{-1}\left[\frac{5}{3s}\right] = \frac{5}{3},$$

since $\mathcal{L}^{-1}[1/s] = 1$.

9. We see that

$$\frac{2}{3s+5} = \frac{2}{3}\cdot\frac{1}{s+5/3},$$

so

$$\mathcal{L}^{-1}\left[\frac{2}{3s+5}\right] = \frac{2}{3}e^{-\frac{5}{3}t}.$$

10. Using the method of partial fractions,

$$\frac{14}{(3s+2)(s-4)} = \frac{A}{3s+2} + \frac{B}{s-4}.$$

Putting the right-hand side over a common denominator gives $A(s-4)+B(3s+2) = 14$, which can be written as $(A+3B)s + (-4A+2B) = 14$. So, $A+3B = 0$ and $-4A+2B = 14$. Thus $A = -3$ and $B = 1$, and

$$\mathcal{L}^{-1}\left[\frac{14}{(3s+2)(s-4)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s-4} - \frac{3}{3s+2}\right].$$

Finally,

$$\mathcal{L}^{-1}\left[\frac{5}{(s-1)(s-2)}\right] = e^{4t} - e^{-2t/3}.$$

11. Using the method of partial fractions, we write

$$\frac{4}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}.$$

Putting the right-hand side over a common denominator gives $A(s+3) + B(s) = 4$, which can be written as $(A+B)s + 3A = 4$. Thus, $A+B = 0$, and $3A = 4$. This gives $A = 4/3$ and $B = -4/3$, so

$$\mathcal{L}^{-1}\left[\frac{4}{s(s+3)}\right] = \mathcal{L}^{-1}\left[\frac{4/3}{s} - \frac{4/3}{s+3}\right].$$

Hence,

$$\mathcal{L}^{-1}\left[\frac{4}{s(s+3)}\right] = \frac{4}{3} - \frac{4}{3}e^{-3t}.$$

12. Using the method of partial fractions, we write

$$\frac{5}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}.$$

Putting the right-hand side over a common denominator gives $A(s-2) + B(s-1) = 5$, which can be written as $(A+B)s + (-2A-B) = 5$. Thus, $A+B=0$, and $-2A-B=5$. This gives $A=-5$ and $B=5$, so

$$\mathcal{L}^{-1}\left[\frac{5}{(s-1)(s-2)}\right] = \mathcal{L}^{-1}\left[\frac{5}{s-2} - \frac{5}{s-1}\right].$$

Thus,

$$\mathcal{L}^{-1}\left[\frac{5}{(s-1)(s-2)}\right] = 5e^{2t} - 5e^{t}.$$

13. Using the method of partial fractions, we have

$$\frac{2s+1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}.$$

Putting the right-hand side over a common denominator gives $A(s-2) + B(s-1) = 2s+1$, which can be written as $(A+B)s + (-2A-B) = 2s+1$. So, $A+B=2$, and $-2A-B=1$. Thus $A=-3$ and $B=5$, which gives

$$\mathcal{L}^{-1}\left[\frac{2s+1}{(s-1)(s-2)}\right] = \mathcal{L}^{-1}\left[\frac{5}{s-2} - \frac{3}{s-1}\right].$$

Finally,

$$\mathcal{L}^{-1}\left[\frac{2s+1}{(s-1)(s-2)}\right] = 5e^{2t} - 3e^{t}.$$

14. Using the method of partial fractions,

$$\frac{2s^2+3s-2}{s(s+1)(s-2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-2}.$$

Putting the right-hand side over a common denominator gives

$$A(s+1)(s-2) + Bs(s-2) + Cs(s+1) = 2s^2+3s-2,$$

which can be written as $(A+B+C)s^2 + (-A-2B+C)s - 2A = 2s^2+3s-2$. So, $A+B+C=2$, $-A-2B+C=3$, and $-2A=-2$. Thus $A=1$, $B=-1$, and $C=2$, and

$$\mathcal{L}^{-1}\left[\frac{2s^2+3s-2}{s(s+1)(s-2)}\right] = \mathcal{L}^{-1}\left[\frac{2}{s-2} - \frac{1}{s+1} + \frac{1}{s}\right].$$

Hence,

$$\mathcal{L}^{-1}\left[\frac{2s^2+3s-2}{s(s+1)(s-2)}\right] = 2e^{2t} - e^{-t} + 1.$$

15. **(a)** We have

$$\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$$

and

$$\mathcal{L}[-y + e^{-2t}] = \mathcal{L}[-y] + \mathcal{L}[e^{-2t}] = -\mathcal{L}[y] + \frac{1}{s+2}$$

using linearity of the Laplace transform and the formula $\mathcal{L}[e^{at}] = 1/(s-a)$ from the text.

(b) Substituting the initial condition yields

$$s\mathcal{L}[y] - 2 = -\mathcal{L}[y] + \frac{1}{s+2}$$

so that

$$(s+1)\mathcal{L}[y] = 2 + \frac{1}{s+2}$$

which gives

$$\mathcal{L}[y] = \frac{1}{(s+1)(s+2)} + \frac{2}{s+1} = \frac{2s+5}{(s+1)(s+2)}.$$

(c) Using the method of partial fractions,

$$\frac{2s+5}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}.$$

Putting the right-hand side over a common denominator gives $A(s+2) + B(s+1) = 2s+5$, which can be written as $(A+B)s + (2A+B) = 2s+5$. So we have $A+B = 2$, and $2A+B = 5$. Thus, $A = 3$ and $B = -1$, and

$$\mathcal{L}[y] = \frac{3}{s+1} - \frac{1}{s+2}.$$

Therefore, $y(t) = 3e^{-t} - e^{-2t}$ is the desired function.

(d) Since $y(0) = 3e^0 - e^0 = 2$, $y(t)$ satisfies the given initial condition. Also,

$$\frac{dy}{dt} = -3e^{-t} + 2e^{-2t}$$

and

$$-y + e^{-2t} = -3e^{-t} + e^{-2t} + e^{-2t} = -3e^{-t} + 2e^{-2t},$$

so our solution also satisfies the differential equation.

16. **(a)** Taking Laplace transforms of both sides of the equation and simplifying gives

$$\mathcal{L}\left[\frac{dy}{dt}\right] + 5\mathcal{L}[y] = \mathcal{L}[e^{-t}]$$

so

$$s\mathcal{L}[y] - y(0) + 5\mathcal{L}[y] = \frac{1}{s+1}$$

and $y(0) = 2$ gives

$$s\mathcal{L}[y] - 2 + 5\mathcal{L}[y] = \frac{1}{s+1}.$$

(b) Solving for $\mathcal{L}[y]$ gives

$$\mathcal{L}[y] = \frac{2}{s+5} + \frac{1}{(s+5)(s+1)} = \frac{2s+3}{(s+5)(s+1)}.$$

(c) Using the method of partial fractions,

$$\frac{2s+3}{(s+5)(s+1)} = \frac{A}{s+5} + \frac{B}{s+1}.$$

Putting the right-hand side over a common denominator gives $A(s+1) + B(s+5) = 2s+3$, which can be written as $(A+B)s + (A+5B) = 2s+3$. So $A+B=2$, and $A+5B=3$. Hence, $A = 7/4$ and $B = 1/4$ and we have

$$\mathcal{L}[y] = \frac{7/4}{s+5} + \frac{1/4}{s+1}.$$

Therefore,

$$y(t) = \frac{7}{4}e^{-5t} + \frac{1}{4}e^{-t}.$$

(d) To check, compute

$$\frac{dy}{dt} + 5y = -\frac{35}{4}e^{-5t} - \frac{1}{4}e^{-t} + 5\left(\frac{7}{4}e^{-5t} + \frac{1}{4}e^{-t}\right) = e^{-t},$$

and $y(0) = 7/4 + 1/4 = 2$.

17. **(a)** Taking Laplace transforms of both sides of the equation and simplifying gives

$$\mathcal{L}\left[\frac{dy}{dt}\right] + 7\mathcal{L}[y] = \mathcal{L}[1]$$

so

$$s\mathcal{L}[y] - y(0) + 7\mathcal{L}[y] = \frac{1}{s}$$

and $y(0) = 3$ gives

$$s\mathcal{L}[y] - 3 + 7\mathcal{L}[y] = \frac{1}{s}.$$

(b) Solving for $\mathcal{L}[y]$ gives

$$\mathcal{L}[y] = \frac{3}{s+7} + \frac{1}{s(s+7)} = \frac{3s+1}{s(s+7)}.$$

(c) Using the method of partial fractions, we get

$$\frac{3s+1}{s(s+7)} = \frac{A}{s} + \frac{B}{s+7}.$$

Putting the right-hand side over a common denominator gives $A(s+7) + Bs = 3s+1$, which can be written as $(A+B)s + 7A = 3s+1$. So $A+B=3$, and $7A=1$. Hence, $A = 1/7$ and $B = 20/7$, and we have

$$\mathcal{L}[y] = \frac{1/7}{s} + \frac{20/7}{s+7}.$$

Thus,

$$y(t) = \frac{20}{7}e^{-7t} + \frac{1}{7}.$$

(d) To check, we compute

$$\frac{dy}{dt} + 7y = -20e^{-7t} + 7\left(\frac{20}{7}e^{-7t} + \frac{1}{7}\right) = 1,$$

and $y(0) = 20/7 + 1/7 = 3$, so our solution satisfies the initial-value problem.

18. **(a)** Taking Laplace transforms of both sides of the equation and simplifying gives

$$\mathcal{L}\left[\frac{dy}{dt}\right] + 4\mathcal{L}[y] = \mathcal{L}[6]$$

so

$$s\mathcal{L}[y] - y(0) + 4\mathcal{L}[y] = \frac{6}{s}$$

and $y(0) = 0$ gives

$$s\mathcal{L}[y] + 4\mathcal{L}[y] = \frac{6}{s}.$$

(b) Solving for $\mathcal{L}[y]$ gives

$$\mathcal{L}[y] = \frac{6}{s(s+4)}.$$

(c) Using the method of partial fractions,

$$\frac{6}{s(s+4)} = \frac{A}{s} + \frac{B}{s+4}.$$

Putting the right-hand side over a common denominator gives $A(s+4) + Bs = 6$, which can be written as $(A+B)s + 4A = 6$. So, $A + B = 0$, and $4A = 6$. Hence, $A = 3/2$ and $B = -3/2$, and we have

$$\mathcal{L}[y] = \frac{3/2}{s} - \frac{3/2}{s+4}.$$

Thus,

$$y(t) = \frac{3}{2} - \frac{3}{2}e^{-4t}.$$

(d) To check, we compute

$$\frac{dy}{dt} + 4y = 6e^{-4t} + 4\left(\frac{3}{2} - \frac{3}{2}e^{-4t}\right) = 6,$$

and $y(0) = 3/2 - 3/2 = 0$, so our solution satisfies the initial-value problem.

19. **(a)** Taking Laplace transforms of both sides of the equation and simplifying gives

$$\mathcal{L}\left[\frac{dy}{dt}\right] + 9\mathcal{L}[y] = \mathcal{L}[2]$$

so

$$s\mathcal{L}[y] - y(0) + 9\mathcal{L}[y] = \frac{2}{s}$$

and $y(0) = -2$ gives

$$s\mathcal{L}[y] + 2 + 9\mathcal{L}[y] = \frac{2}{s}.$$