2. (a) If $H(x, y) = \sin(xy)$, then

$$\frac{\partial H}{\partial x} = y\cos(xy)$$

and so

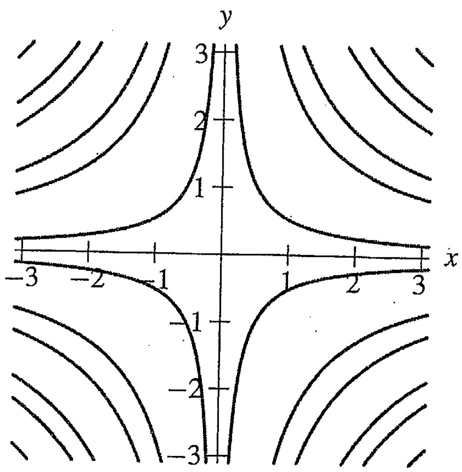
$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}.$$

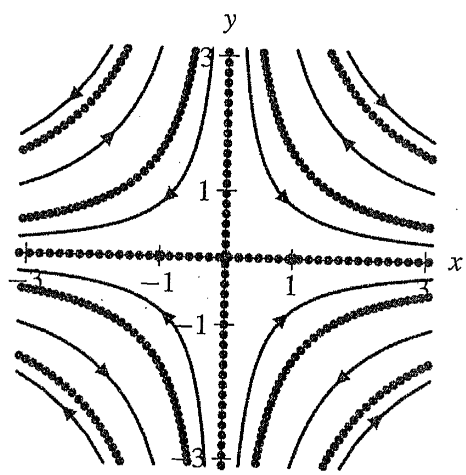
Similarly,

$$\frac{\partial H}{\partial y} = x\cos(xy) = \frac{dx}{dt}.$$

(b) Note that the level sets of H are the same curves as those of the level sets of xy.

(c) Note that there are many curves of equilibrium points for this system: besides the origin, whenever $xy = n\pi + \pi/2$, the vector field vanishes.





3. (a) If $H(x, y) = x\cos y + y^2$, then

$$\frac{\partial H}{\partial x} = \cos y$$

and so

$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}.$$

Similarly,

$$\frac{\partial H}{\partial y} = -x\sin y + 2y = \frac{dx}{dt}.$$

If we differentiate $H(x, y)$ with respect to x, we get

$$y + c'(x),$$

which we want to be the negative of $dy/dt = -y$. Hence $c'(x) = 0$, and we pick the antiderivative $c(x) = 0$. A Hamiltonian function is

$$H(x, y) = xy - y^3.$$

12. First we check to see if the partial derivative with respect to x of the first component of the vector field is the negative of the partial derivative with respect to y of the second component. We have

$$\frac{\partial 1}{\partial x} = 0$$

while

$$-\frac{\partial y}{\partial y} = -1.$$

Since these are not equal, the system is not Hamiltonian.

13. First we check to see if the partial derivative with respect to x of the first component of the vector field is the negative of the partial derivative with respect to y of the second component. We have

$$\frac{\partial (x \cos y)}{\partial x} = \cos y$$

while

$$-\frac{\partial (-y \cos x)}{\partial y} = \cos x.$$

Since these two are not equal, the system is not Hamiltonian.

14. First note that

$$\frac{\partial F(y)}{\partial x} = 0 = -\frac{\partial G(x)}{\partial y},$$

that is, the partial derivative of the x component of the vector field with respect to x is equal to the negative of the partial derivative of the y component with respect to y. Hence, the system is Hamiltonian. Integrating the x component of the vector field with respect to y yields

$$H(x, y) = \int F(y)\, dy + c$$

where the "constant" c could depend on x. If we differentiate this H with respect to x we get

$$-\frac{\partial H}{\partial x} = -c'(x).$$

Thus we take $c = -\int G(x)\, dx$. A Hamiltonian function is

$$H(x, y) = \int F(y)\, dy - \int G(x)\, dx.$$

16. **(a)** We first check

$$\frac{\partial(-yx^2)}{\partial x} = -2xy \neq -\frac{\partial(x+1)}{\partial y} = 0,$$

so the system is not Hamiltonian.

(b) If we multiply the vector field by $1/x^2$, we obtain the new system

$$\frac{dx}{dt} = -y$$
$$\frac{dy}{dt} = \frac{1}{x} + \frac{1}{x^2}.$$

As in Exercise 14, this system is Hamiltonian with

$$H(x, y) = \frac{1}{x} - \ln|x| - \frac{y^2}{2}.$$

17. Using the technique of Exercise 15, we we multiply the vector field by $1/(2-y)$. As in Exercise 14, the resulting system

$$\frac{dx}{dt} = \frac{1-y^2}{2-y}$$
$$\frac{dy}{dt} = x$$

is Hamiltonian. The Hamiltonian is

$$\begin{aligned} H(x, y) &= -\frac{x^2}{2} + \int \frac{y^2-1}{y-2}\, dy \\ &= -\frac{x^2}{2} + \int 2 + y + \frac{3}{y-2}\, dy \\ &= -\frac{x^2}{2} + 2y + \frac{y^2}{2} + 3\ln|y-2|. \end{aligned}$$

The function

$$H(x, y) = -\frac{x^2}{2} + 2y + \frac{y^2}{2} + 3\ln|y-2|$$

is a conserved quantity for the original system. However, it is not defined on the line $y = 2$. From the system, we see that this line is a single solution curve that separates the two half-planes, $y < 2$ and $y > 2$.

18. **(a)** We have

$$\frac{\partial H}{\partial y} = y \quad \text{and} \quad -\frac{\partial H}{\partial x} = x^2 - a,$$

so this system is Hamiltonian with the given function H.

(b) Note that $dx/dt = 0$ if and only if $y = 0$ and $dy/dt = 0$ if and only if $x = \pm\sqrt{a}$. Consequently if $a < 0$, then there are no equilibrium points. If $a = 0$, there is one equilibrium point at $(0, 0)$ and if $a > 0$, there are two equilibrium points at $(\pm\sqrt{a}, 0)$.

(c) The Jacobian matrix is

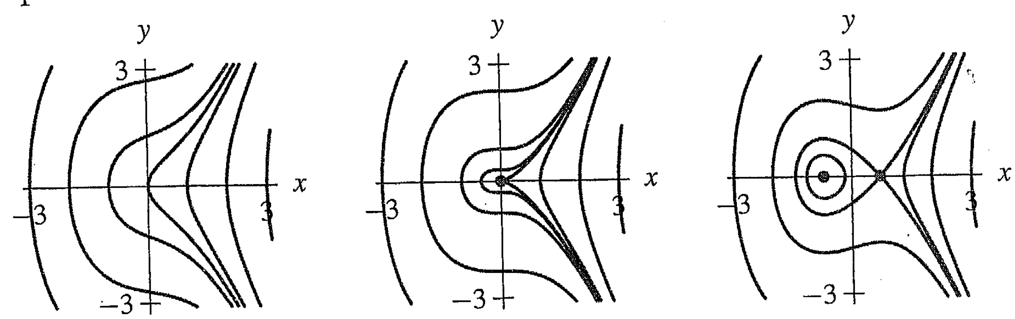
$$\begin{pmatrix} 0 & 1 \\ 2x & 0 \end{pmatrix},$$

which, when evaluated at the equilibrium points, becomes

$$\begin{pmatrix} 0 & 1 \\ \pm 2\sqrt{a} & 0 \end{pmatrix}.$$

At $(\sqrt{a}, 0)$, the eigenvalues are $\pm\sqrt{2\sqrt{a}}$ so this equilibrium point is a saddle. At $(-\sqrt{a}, 0)$, the eigenvalues are $\pm i\sqrt{2\sqrt{a}}$ so this equilibrium point is a center. If $a = 0$ the eigenvalues are both 0, so this point is a node.

(d)



Phase portrait for $a < 0$ Phase portrait for $a = 0$ Phase portrait for $a > 0$

(e) As a increases toward 0, the phase portrait changes from having no equilibrium points to having a single equilibrium point at $a = 0$. If $a > 0$, there is a pair of equilibrium points.

19. First note that this system is Hamiltonian for every value of a. The Hamiltonian function depends on a and is given by

$$H(x, y) = x^2y + xy^2 - ax.$$

If $a > 0$, then the system has two saddle equilibrium points on the y-axis at $(0, \pm\sqrt{a})$. If $a = 0$, then system has only one equilibrium point at $(0, 0)$. If $a < 0$, the system again has two saddles, but they are now located at $(\pm 2\sqrt{-3a}/3, \mp\sqrt{-3a}/3)$. This corresponds to a change in shape of the graph of H.

20. (a) First note that the system is still Hamiltonian, with Hamiltonian function

$$H(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3 + ax.$$

The equilibrium points are

$$\left(\frac{1 \pm \sqrt{1 - 4a}}{2}, 0\right).$$

Hence there are no equilibrium points if $a > 1/4$; one equilibrium point if $a = 1/4$; and two equilibrium points if $a < 1/4$. A bifurcation occurs at $a = 1/4$.

(b) The book would never have appeared. Wouldn't that have been awful?