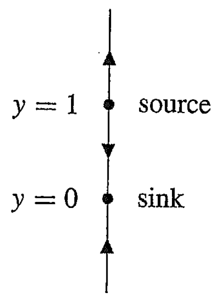
EXERCISES FOR SECTION 1.6

1. The equilibrium points of $dy/dt = f(y)$ are the numbers y where $f(y) = 0$. For

$$f(y) = 3y(y-1),$$

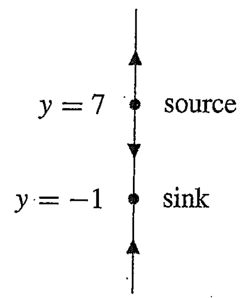
the equilibrium points are $y = 0$ and $y = 1$. Since $f(y)$ is positive for $y < 0$, negative for $0 < y < 1$, and positive for $y > 1$, the equilibrium point $y = 0$ is a sink and the equilibrium point $y = 1$ is a source.



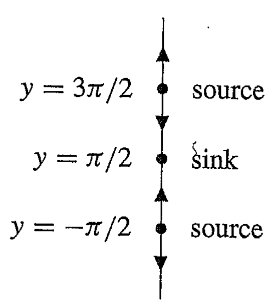
2. The equilibrium points of $dy/dt = f(y)$ are the numbers y where $f(y) = 0$. For

$$f(y) = y^2 - 6y - 7 = (y-7)(y+1),$$

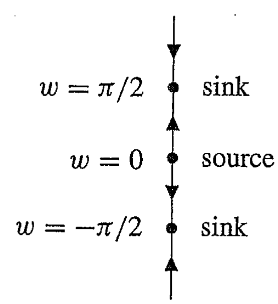
the equilibrium points are $y = -1$ and $y = 7$. Since $f(y)$ is positive for $y < -1$, negative for $-1 < y < 7$, and positive for $y > 7$, the equilibrium point $y = -1$ is a sink and the equilibrium point $y = 7$ is a source.



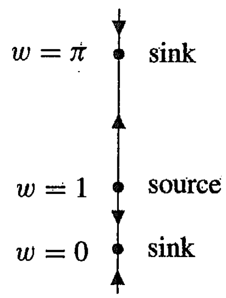
3. The equilibrium points of $dy/dt = f(y)$ are the numbers y where $f(y) = 0$. For $f(y) = \cos y$, the equilibrium points are $y = \pi/2 + n\pi$, where $n = 0, \pm1, \pm2, \ldots$. Since $\cos y > 0$ for $-\pi/2 < y < \pi/2$ and $\cos y < 0$ for $\pi/2 < y < 3\pi/2$, we see that the equilibrium point at $y = \pi/2$ is a sink. Since the sign of $\cos y$ alternates between positive and negative in a period fashion, we see that the equilibrium points at $y = \pi/2 + 2n\pi$ are sinks and the equilibrium points at $y = 3\pi/2 + 2n\pi$ are sources.



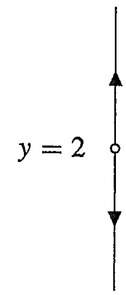
4. The equilibrium points of $dw/dt = f(w)$ are the numbers w where $f(w) = 0$. For $f(w) = w \cos w$, the equilibrium points are $w = 0$ and $w = \pi/2 + n\pi$, where $n = 0, \pm1, \pm2, \ldots$. The sign of $w \cos w$ alternates positive and negative at successive zeros. It is negative for $-\pi/2 < w < 0$ and positive for $0 < w < \pi/2$. Therefore, $w = 0$ is a source, and the equilibrium points alternate back and forth between sources and sinks.



5. The equilibrium points of $dw/dt = f(w)$ are the numbers w where $f(w) = 0$. For $f(w) = (w-1)\sin w$, the equilibrium points are $w = 1$ and $w = n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$. The sign of $(w-1)\sin w$ alternates between positive and negative at successive zeros. It is positive for $-\pi < w < 0$ and negative for $0 < w < 1$. Therefore, $w = 0$ is a sink, and the equilibrium points alternate between sources and sinks.

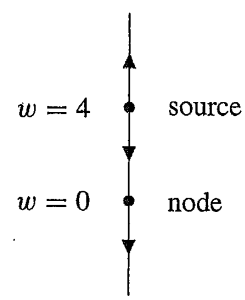


6. This equation has no equilibrium points, but the equation is not defined at $y = 2$. For $y > 2$, $dy/dt > 0$, so solutions increase. If $y < 2$, $dy/dt < 0$, so solutions decrease. The solutions approach the point $y = 2$ as time decreases and actually arrive there in finite time.



7. The derivative dv/dt is always positive, so there are no equilibrium points, and all solutions are increasing.

8. The equilibrium points of $dw/dt = f(w)$ are the numbers w where $f(w) = 0$. For $f(w) = 3w^3 - 12w^2$, the equilibrium points are $w = 0$ and $w = 4$. Since $f(w) < 0$ for $w < 0$ and $0 < w < 4$, and $f(w) > 0$ for $w > 4$, the equilibrium point at $w = 0$ is a node and the equilibrium point at $w = 4$ is a source.



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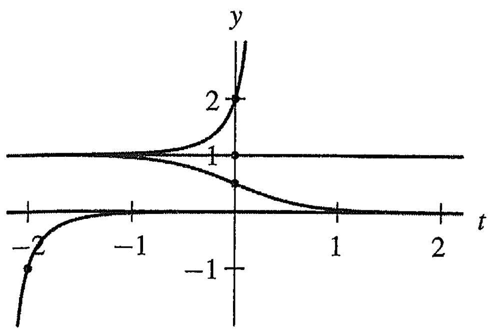

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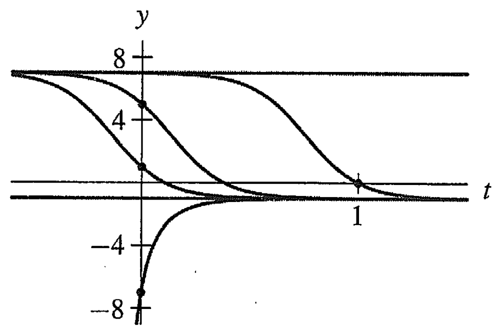

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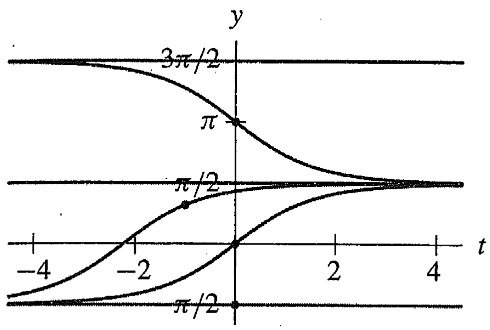

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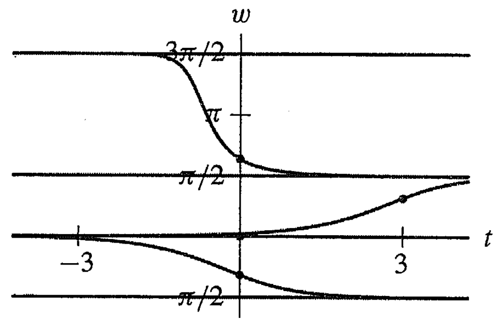

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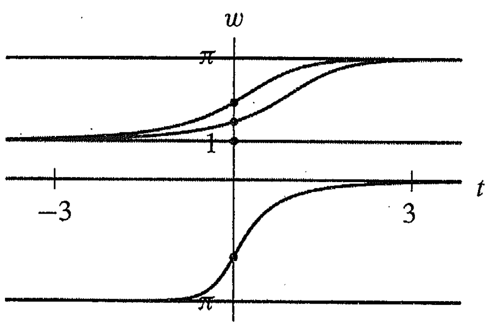

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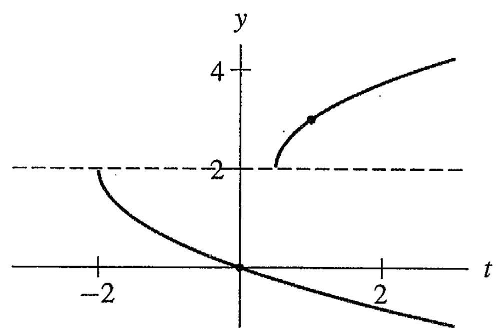
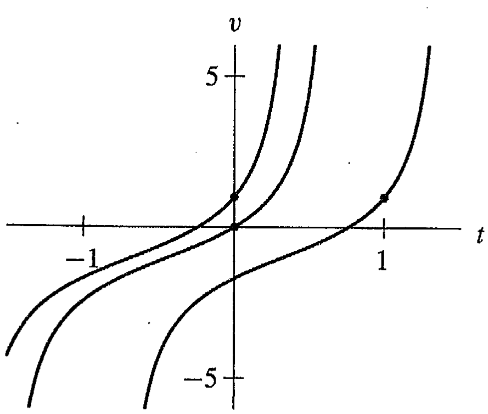


The equation is undefined at $y = 2$.

19.



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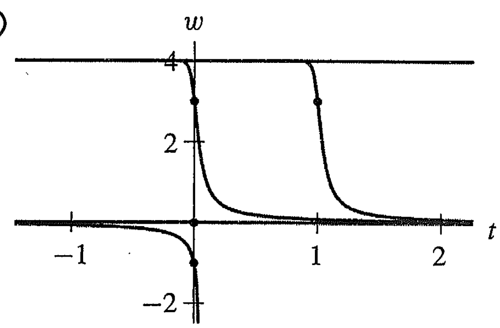


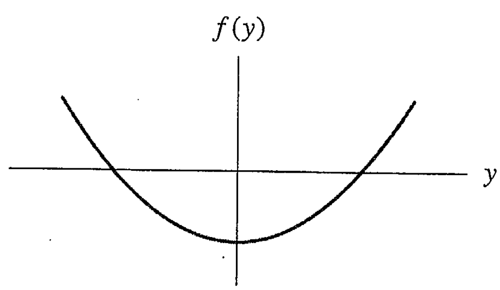
30. The function $f(y)$ has three zeros. We denote them as y_1, y_2, and y_3, where $y_1 < 0 < y_2 < y_3$. So the differential equation $dy/dt = f(y)$ has three equilibrium solutions, one for each zero. Also, $f(y) > 0$ if $y < y_1$, $f(y) < 0$ if $y_1 < y < y_2$, and $f(y) > 0$ if $y_2 < y < y_3$ or if $y > y_3$. Hence y_1 is a sink, y_2 is a source, and y_3 is a node.

31. The function $f(y)$ has two zeros, one positive and one negative. We denote them as y_1 and y_2, where $y_1 < y_2$. So the differential equation $dy/dt = f(y)$ has two equilibrium solutions, one for each zero. Also, $f(y) > 0$ if $y_1 < y < y_2$ and $f(y) < 0$ if $y < y_1$ or if $y > y_2$. Hence y_1 is a source and y_2 is a sink.

32. The function $f(y)$ has four zeros, which we denote $y_1, \ldots, y_4$ where $y_1 < 0 < y_2 < y_3 < y_4$. So the differential equation $dy/dt = f(y)$ has four equilibrium solutions, one for each zero. Also, $f(y) > 0$ if $y < y_1$, if $y_2 < y < y_3$, or if $y_3 < y < y_4$; and $f(y) < 0$ if $y_1 < y < y_2$ or if $y > y_4$. Hence y_1 is a sink, y_2 is a source, y_3 is a node, and y_4 is a sink.

33. Since there are two equilibrium points, the graph of $f(y)$ must touch the y-axis at two distinct numbers y_1 and y_2. Assume that $y_1 < y_2$. Since the arrows point up if $y < y_1$ and if $y > y_2$, we must have $f(y) > 0$ for $y < y_1$ and for $y > y_2$. Similarly, $f(y) < 0$ for $y_1 < y < y_2$.

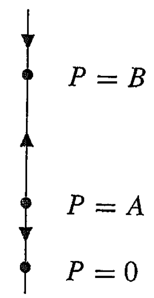
The precise location of the equilibrium points is not given, and the direction of the arrows on the phase line is determined only by the sign (and not the magnitude) of $f(y)$. So the following graph is one of many possible answers.



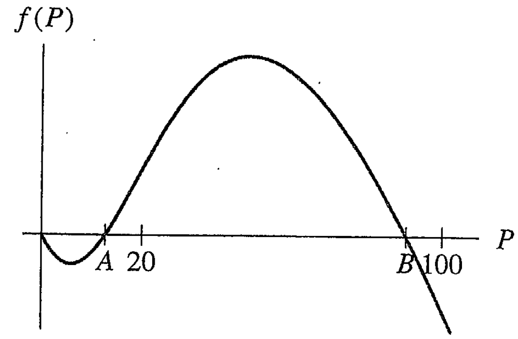
34. Since there are three equilibrium points (one appearing to be at $y = 0$), the graph of $f(y)$ must touch the y-axis at three numbers y_1, y_2, and y_3. We assume that $y_1 < y_2 = 0 < y_3$. Since the arrows point down for $y < y_1$ and $y_2 < y < y_3$, $f(y) < 0$ for $y < y_1$ and for $y_2 < y < y_3$. Similarly, $f(y) > 0$ if $y_1 < y < y_2$ and if $y > y_3$.

The precise location of the equilibrium points is not given, and the direction of the arrows on the phase line is determined only by the sign (and not the magnitude) of $f(y)$. So the following graph is one of many possible answers.

40. **(a)** From the information given, there is an equilibrium point at $P = 0$ and at least two more equilibrium points, $P = A$ and $P = B$, where $0 < A < 20$ and $20 < B < 100$. The following phase line is the simplest one that is consistent with the assumptions.



(b) The following graph is one of many possible answers.

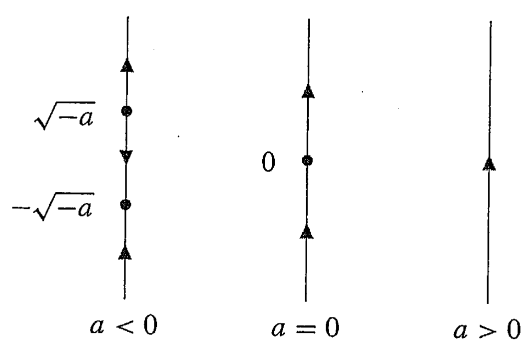


(c) There could be many more equilibrium points between $P = 20$ and $P = 100$ and between $P = 0$ and $P = 20$.

41. The equilibrium points occur at solutions of $dy/dt = y^2 + a = 0$. For $a > 0$, there are no equilibrium points. For $a = 0$, there is one equilibrium point, $y = 0$. For $a < 0$, there are two equilibrium points, $y = \pm\sqrt{-a}$.

To draw the phase lines, note that:

- If $a > 0$, $dy/dt = y^2 + a > 0$, so the solutions are always increasing.
- If $a = 0$, $dy/dt > 0$ unless $y = 0$. Thus, $y = 0$ is a node.
- For $a < 0$, $dy/dt < 0$ for $-\sqrt{-a} < y < \sqrt{-a}$, and $dy/dt > 0$ for $y < -\sqrt{-a}$ and for $y > \sqrt{-a}$.



(a) The phase lines for $a < 0$ are qualitatively the same, and the phase lines for $a > 0$ are qualitatively the same.

(b) The phase line undergoes a qualitative change at $a = 0$.

42. The equilibrium points occur at solutions of $dy/dt = ay - y^3 = 0$. For $a \leq 0$, there is one equilibrium point, $y = 0$. For $a > 0$, there are three equilibrium points, $y = 0$ and $y = \pm\sqrt{a}$.

To draw the phase lines, note that: