(b) The independent variable is t, and x and y are the dependent variables. Since both xy-terms are negative, these species compete. The parameter γ is the growth-rate coefficient for x, and α is the growth-rate parameter for y. Neither population has a carrying capacity. The parameter δ measures the harm to x caused by the interaction of the two species, and β measures the harm to y caused by the interaction.

22. **(a)** The effect of y on dx/dt is through the term $-by\sqrt{x}$. Because this term is negative (x and y are positive), a larger y population decreases dx/dt, so x is the prey. Similarly, the term $cy\sqrt{x}$ gives the effect of x on dy/dt and a larger x gives a larger dy/dt. So y is the predator.

(b) If $x = 0$, $dy/dt = 0$, so the predator population is constant. The predator must have alternate sources of food or be able to hibernate without need for prey.

XERCISES FOR SECTION 1.2

1. **(a)** Let's check Bob's solution first. Since $dy/dt = 1$ and

$$\frac{y(t)+1}{t+1} = \frac{t+1}{t+1} = 1,$$

Bob's answer is correct.

Now let's check Glen's solution. Since $dy/dt = 2$ and

$$\frac{y(t)+1}{t+1} = \frac{2t+2}{t+1} = 2,$$

Glen's solution is also correct.

Finally let's check Paul's solution. We have $dy/dt = 2t$ on one hand and

$$\frac{y(t)+1}{t+1} = \frac{t^2-1}{t+1} = t-1$$

on the other. Paul is wrong.

(b) At first glance, they should have seen the equilibrium solution $y(t) = -1$ for all t because $dy/dt = 0$ for any constant function and $y = -1$ implies that

$$\frac{y+1}{t+1} = 0$$

independent of t.

Strictly speaking the differential equation is not defined for $t = -1$, and hence the solutions are not defined for $t = -1$.

2. We note that $dy/dt = 2e^{2t}$ for $y(t) = e^{2t}$. If $y(t) = e^{2t}$ is a solution to the differential equation, then we must have

$$\begin{aligned} 2e^{2t} &= 2y(t) - t + g(y(t)) \\ &= 2e^{2t} - t + g(e^{2t}). \end{aligned}$$

Hence, we need

$$g(e^{2t}) = t.$$

This equation is satisfied if we let $g(y) = (\ln y)/2$. In other words, $y(t) = e^{2t}$ is a solution of the differential equation

$$\frac{dy}{dt} = 2y - t + \frac{\ln y}{2}.$$

3. In order to find one such $f(t, y)$, we compute the derivative of $y(t)$. We obtain

$$\frac{dy}{dt} = \frac{de^{t^3}}{dt} = 3t^2 e^{t^3}.$$

Now we replace e^{t^3} in the last expression by y and get the differential equation

$$\frac{dy}{dt} = 3t^2 y.$$

4. Starting with $dP/dt = kP$, we divide both sides by P to obtain

$$\frac{1}{P}\frac{dP}{dt} = k.$$

Then integrating both sides with respect to t, we have

$$\int \frac{1}{P}\frac{dP}{dt}\, dt = \int k\, dt,$$

and changing variables on the left-hand side, we obtain

$$\int \frac{1}{P}\, dP = \int k\, dt.$$

(Typically, we jump to the equation above by "informally" multiplying both sides by dt.) Integrating, we get

$$\ln |P| = kt + c,$$

where c is an arbitrary constant. Exponentiating both sides gives

$$|P| = e^{kt+c} = e^c e^{kt}.$$

For population models we consider only $P \geq 0$, and the absolute value sign is unnecessary. Letting $P_0 = e^c$, we have

$$P(t) = P_0 e^{kt}.$$

In general, it is possible for $P(0)$ to be negative. In that case, $e^c = -P_0$, and $|P| = -P$. Once again we obtain

$$P(t) = P_0 e^{kt}.$$

5. The constant function $y(t) = 0$ is an equilibrium solution.

For $y \neq 0$ we separate the variables and integrate

$$\int \frac{dy}{y} = \int t\,dt$$

$$\ln|y| = \frac{t^2}{2} + c$$

$$|y| = c_1 e^{t^2/2}$$

where $c_1 = e^c$ is an arbitrary positive constant.

If $y > 0$, then $|y| = y$ and we can just drop the absolute value signs in this calculation. If $y < 0$, then $|y| = -y$, so $-y = c_1 e^{t^2/2}$. Hence, $y = -c_1 e^{t^2/2}$. Therefore,

$$y = ke^{t^2/2}$$

where $k = \pm c_1$. Moreover, if $k = 0$, we get the equilibrium solution. Thus, $y = ke^{t^2/2}$ yields all solutions to the differential equation if we let k be any real number. (Strickly speaking we need a theorem from Section 1.5 to justify the assertion that this formula provides all solutions.)

6. Separating variables and integrating, we obtain

$$\int \frac{1}{y}\,dy = \int t^4\,dt$$

$$\ln|y| = \frac{t^5}{5} + c$$

$$|y| = c_1 e^{t^5/5},$$

where $c_1 = e^c$. As in Exercise 5, we can eliminate the absolute values by replacing the positive constant c_1 with $k = \pm c_1$. Hence, the general solution is

$$y(t) = ke^{t^5/5},$$

where k is any real number. Note that $k = 0$ gives the equilibrium solution.

7. We separate variables and integrate to obtain

$$\int \frac{dy}{2y+1} = \int dt.$$

We get

$$\frac{1}{2}\ln|2y+1| = t + c$$

$$|2y+1| = c_1 e^{2t},$$

where $c_1 = e^{2c}$. As in Exercise 5, we can drop the absolute value signs by replacing $\pm c_1$ with a new constant k_1. Hence, we have

$$2y + 1 = k_1 e^{2t}$$

$$y = \frac{1}{2}\left(k_1 e^{2t} - 1\right),$$

and letting $k = k_1/2$, $y(t) = ke^{2t} - 1/2$. Note that, for $k = 0$, we get the equilibrium solution.

24. First we find the general solution by writing the differential equation as

$$\frac{dy}{dt} = (t+2)y^2,$$

separating variables, and integrating. We have

$$\int \frac{1}{y^2}\,dy = \int (t+2)\,dt$$

$$-\frac{1}{y} = \frac{t^2}{2} + 2t + c$$

$$= \frac{t^2 + 4t + c_1}{2},$$

where $c_1 = 2c$. Inverting and multiplying by -1 produces

$$y(t) = \frac{-2}{t^2 + 4t + c_1}.$$

Setting

$$1 = y(0) = \frac{-2}{c_1}$$

and solving for c_1, we obtain $c_1 = -2$. So

$$y(t) = \frac{-2}{t^2 + 4t - 2}.$$

25. Separating variables and integrating, we obtain

$$\int \frac{dy}{y^2} = -\int dt$$

$$-\frac{1}{y} = -t + c.$$

So we get

$$y = \frac{1}{t - c}.$$

Now we need to find the constant c so that $y(0) = 1/2$. To do this we solve

$$\frac{1}{2} = \frac{1}{0 - c}$$

and get $c = -2$. The solution of the initial-value problem is

$$y(t) = \frac{1}{t+2}.$$

26. First we separate variables and integrate to obtain

$$\int y^{-3}\,dy = \int t^2\,dt,$$

which yields

$$-\frac{y^{-2}}{2} = \frac{t^3}{3} + c.$$

Solving for y gives

$$y^2 = \frac{1}{c_1 - 2t^3/3},$$

where $c_1 = -2c$. So

$$y(t) = \pm\frac{1}{\sqrt{c_1 - 2t^3/3}}.$$

The initial value $y(0)$ is negative, so we choose the negative square root and obtain

$$y(t) = -\frac{1}{\sqrt{c_1 - 2t^3/3}}.$$

Using $-1 = y(0) = -1/\sqrt{c_1}$, we see that $c_1 = 1$ and the solution of the initial-value problem is

$$y(t) = -\frac{1}{\sqrt{1 - 2t^3/3}}.$$

27. We do not need to do any computations to solve this initial-value problem. We know that the constant function $y(t) = 0$ for all t is an equilibrium solution, and it satisfies the initial condition.

28. Rewriting the equation as

$$\frac{dy}{dt} = \frac{t}{(1-t^2)y},$$

we separate variables and integrate obtaining

$$\int y\,dy = \int \frac{t}{1-t^2}\,dt$$

$$\frac{y^2}{2} = -\frac{1}{2}\ln|1-t^2| + c$$

$$y = \pm\sqrt{-\ln|1-t^2| + k}.$$

Since $y(0) = 4$ is positive, we use the positive square root and solve

$$4 = y(0) = \sqrt{-\ln|1| + k} = \sqrt{k}$$

for k. We obtain $k = 16$. Hence,

$$y(t) = \sqrt{16 - \ln(1-t^2)}.$$

We may replace $|1-t^2|$ with $(1-t^2)$ because the solution is only defined for $-1 < t < 1$.

31. We separate variables to obtain

$$\int \frac{dy}{1+y^2} = \int t\,dt$$

$$\arctan y = \frac{t^2}{2} + c,$$

where c is a constant. Hence the general solution is

$$y(t) = \tan\left(\frac{t^2}{2} + c\right).$$

Next we find c so that $y(0) = 1$. Solving

$$1 = \tan\left(\frac{0^2}{2} + c\right)$$

yields $c = \pi/4$, and the solution to the initial-value problem is

$$y(t) = \tan\left(\frac{t^2}{2} + \frac{\pi}{4}\right).$$

32. Separating variables and integrating, we obtain

$$\int (2y+3)\,dy = \int dt$$

$$y^2 + 3y = t + c$$

$$y^2 + 3y - (t + c) = 0.$$

We can use the quadratic formula to obtain

$$y = -\tfrac{3}{2} \pm \sqrt{t + c_1},$$

where $c_1 = c + 9/4$. Since $y(0) = 1 > -3/2$ we take the positive square root and solve

$$1 = y(0) = -\tfrac{3}{2} + \sqrt{c_1},$$

so $c_1 = 25/4$. The solution to the initial-value problem is

$$y(t) = -\tfrac{3}{2} + \sqrt{t + \tfrac{25}{4}}.$$

33. Let $S(t)$ denote the amount of salt (in pounds) in the bucket at time t (in minutes). We derive a differential equation for S by considering the difference between the rate that salt is entering the bucket and the rate that salt is leaving the bucket. Salt is entering the bucket at the rate of 1/4 pounds