

Test 2: Take-Home OR E.C.O.

Math 341 Fall 2010
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DIFFERENTIAL EQUATIONS
DUE: 5 pm, Wednesday December 1st

Name: _____

Directions:

FOR E.C.O. CREDIT you must do the entire test as a take-home exam where the rules are that you can talk about the Exam with any faculty member but NO OTHER STUDENT. **You must submit the completed exam by the due date and receive a score of 90% or better in order to get Extra Credit Opportunity credit.** The educational goal of the exercise is for you to demonstrate mastery of the material that the Exam was on. NOTE: since this is a take-home exam, my standards for EXPLANATIONS and the LEVEL OF DETAIL in the work included will be higher than on the in-class exam.

FOR EXAM CREDIT. The rules of doing the exam are the same as for ECO credit (i.e. it is a take-home). **The score on the take-home exam shall be averaged with your score on the in-class version of the exam.** The educational goal of the exercise is for you to demonstrate mastery of the material that the Exam was on. NOTE: since this is a take-home exam, my standards for EXPLANATIONS and the LEVEL OF DETAIL in the work included will be higher than on the in-class exam. You can talk about the Exam with any faculty member but NO OTHER STUDENT.

Pledge: I, _____, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

CHECK ONE:

ECO CREDIT

EXAM CREDIT

No.	Score	Maximum
1		15
2		15
3		20
BONUS		4
Total		50

1. [15 points total.] **Linear Systems of Differential Equations, Trace-Determinant Plane, Bifurcation.** VISUAL & ANALYTIC.

Consider $\frac{d\vec{x}}{dt} = A\vec{x}$ where $A = \begin{bmatrix} \alpha & -\alpha/2 \\ 1 & -1 \end{bmatrix}$ and α is a known real-valued parameter. Recall

that the eigenvalues of the matrix A are given by $\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$ where T is the trace of the matrix A and D is the determinant of matrix A .

1(a) [5 points]. Compute the trace T and determinant D of matrix A for all values of α . Show that the relationship between the trace T and determinant D is $2D + T = -1$ regardless of the value of α .

1(b) [5 points]. Use your answer from (a) to sketch a graph in the trace-determinant plane depicting the relationship between the trace T and determinant D for the given matrix A as α changes. On the same axes, sketch the standard graph in the trace-determinant plane which separates the occurrence of real eigenvalues from complex eigenvalues for *any* matrix. [HINT: Label your graphs!]

1(c) [5 points]. Does the qualitative nature of the phase portrait (and equilibrium at the origin) change as α varies? If so, **give all the bifurcation values of α , classify the equilibrium point at the origin** for values of α (greater than, less than and equal to the bifurcation value(s)) and **provide reasonable sketches of the phase portrait(s)** in each case in the space below.

2. [15 points total.] Linearization, Hamiltonian function, Gradient function. ANALYTIC, VERBAL & VISUAL.

Are the following statements TRUE or FALSE – put your answer in the box. To receive ANY credit, you must also give a brief, and correct, explanation in support of your answer! For example, if you think the answer is **FALSE** providing a counterexample for which the statement is NOT TRUE is best. If you think the answer is **TRUE** you should prove why you think the statement is always true. Your explanation of your answer is worth FOUR TIMES as much as the answer you put in the box. **For Full Credit you must write a full sentence explaining the reason for your choice of TRUE or FALSE.**

Consider the non-linear system

$$\begin{aligned}\frac{dx}{dt} &= \alpha x - y + y^3 \\ \frac{dy}{dt} &= x + \alpha y + x^2\end{aligned}$$

where α is a known real-valued parameter.

2(a) TRUE or FALSE? “A Hamiltonian function $H(x, y)$ for the given nonlinear system of ODEs exists.”

2(b) TRUE or FALSE? “A gradient function $G(x, y)$ for the nonlinear system of ODEs exists for some value of α .”

2(c) TRUE or FALSE? “The origin of the phase portrait for this nonlinear system will look like a center (a series of concentric circles) for all values of α .”

3. [20 points total.] **Linear Systems of Differential Equations, Matrix Exponential, General Solution.** ANALYTIC & VERBAL.

Consider the initial value problem $\frac{d\vec{x}}{dt} = A\vec{x}$, $\vec{x}(0) = \vec{x}_0$ where $\vec{x} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and $\vec{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

Usually we write the general solution as $\vec{x} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$ where λ_i and \vec{v}_i are eigenvalues and eigenvectors of matrix A . This is called the **general eigenvector solution**. However, the solution can also be written as $\vec{x}(t) = e^{At} \vec{x}_0$, which we will call the **matrix exponential solution**.

The goal of this problem is to show that the general solution to the given initial value problem can be represented using *either* the matrix exponential solution or the general eigenvector solution.

Recall from Calculus that $\frac{d}{dt} e^{\square t} = \square e^{\square t}$ as long as $\frac{d}{dt} \square = 0$.

Recall from Linear Algebra that if A is diagonalizable, then the matrix exponential $e^{At} = S e^{\Lambda t} S^{-1}$ where S is an $n \times n$ matrix whose columns consist of the n eigenvectors $\vec{v}_1, \dots, \vec{v}_n$ of A , and Λ is an $n \times n$ diagonal matrix with the corresponding eigenvalues along the diagonal.

$$e^{At} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \dots & \vec{v}_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 & 0 & 0 \\ 0 & 0 & e^{\lambda_3 t} & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \dots & \vec{v}_n \end{bmatrix}^{-1}$$

NOTE: For this problem you can assume that A is a 2×2 diagonalizable matrix of the form $\begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix}$ (where p and q are known fixed numbers) and A has two eigenvalues λ_1 and λ_2 with associated eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$.

3(a) [3 points]. Show that the **the general eigenvector solution** satisfies the differential equation $\frac{d\vec{x}}{dt} = A\vec{x}$.

3(b) [3 points]. Show that in order for **the general eigenvector solution** to satisfy the given initial condition $\vec{x}(0) = \vec{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, the linear system $\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ must be solved to find the unknown constants c_1 and c_2 . (HINT: What's a restriction involving λ_1 and λ_2 which must be satisfied?)

3(c) [6 points]. Show that **the matrix exponential solution** $\vec{x}(t) = e^{At}\vec{x}_0$ satisfies the given initial value problem. [HINT: for a given solution to satisfy an initial value problem what must be true?]

3(d) [8 points]. Use your results from (a),(b) and (c) to show that **the matrix exponential solution** is identical to **the general eigenvector solution** for the given initial value problem $\frac{d\vec{x}}{dt} = A\vec{x}$, $\vec{x}(0) = \vec{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ where A has two eigenvalues λ_1 and λ_2 with associated eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$. (HINT: What's a restriction involving λ_1 and λ_2 which must be satisfied?) EXPLAIN YOUR ANSWER(S) and SHOW ALL YOUR WORK.

BONUS. [4 points]. Consider the special case from Question #3 where $p = 3$ and $q = 2$ in matrix

A. Find the (matrix exponential) solution to the initial value problem

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and simplify it into the form } \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

CHECK YOUR ANSWER!

$$\text{HINT: } \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$