Test 2: Differential Equations

Math 341 Fall 2008 — Friday November 7

 — 2:30pm-3:25pm

Name: Buckmire

Directions: Read *all* problems first before answering any of them. There are **7** pages in this test. This is a 55-minute closed book, test. **No calculators.** You must show all relevant work to support your answers. Use complete English sentences as much as possible and CLEARLY indicate your final answers to be graded from your "scratch work."

You may use a 8.5" by 11" "cheat sheet" which must be stapled to the exam but it will cost you two (2) BONUS points. (In other words if you do NOT use a cheat sheet, you may earn up to 10 BONUS points, otherwise the BONUS is worth 8 points.)

Pledge: I, ____________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
1		40
2		30
3		30
BONUS		8 or 10
Total		**100**

1. *[40 points total.]* **Linear Systems of Differential Equations.**

Consider the linear system of differential equations $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \vec{x}$ where $\vec{x} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$.

(a) *[10 points].* Write down the corresponding system of ordinary differential equations for the variables $x(t)$, $y(t)$ and $z(t)$.

$$\frac{dx}{dt} = y + z$$

$$\frac{dy}{dt} = x + z$$

$$\frac{dz}{dt} = x + y$$

(b) *[10 points].* Show that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ are **eigenvectors** of $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

What are the corresponding **eigenvalues**? [NOTE: you do not have to compute any determinants to do this problem.]

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow E_2 = \text{span}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = -1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \Rightarrow E_{-1} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right\}$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = -1\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

(c) *[10 points]*. Use your answers in (b) to write down the general solution $\vec{x}(t)$ to the given linear system of ODEs

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \vec{x}.$$

$$\vec{x}(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}$$

$=$

(d) *[10 points]*. CHECK that your general solution given in (c) does indeed solve the given linear system of ODEs.

$$\frac{d\vec{x}}{dt} = 2c_1 e^{2t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - c_2 e^{-t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - c_3\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\vec{x} = c_1 e^{2t}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3 e^{-t}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$= 2c_1 e^{2t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - c_2 e^{-t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - c_3 e^{-t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

equal

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\vec{x}$$

2. *[30 points total.]* **Equilibria of Planar Systems, Hamiltonian, Trace-Determinant Bifurcation**

Are the following statements TRUE or FALSE – put your answer in the box. To receive ANY credit, you must also give a brief, and correct, explanation in support of your answer! For example, if you think the answer is **FALSE** providing a counterexample for which the statement is NOT TRUE is best. If you think the answer is **TRUE** you should prove why you think the statement is always true. Your explanation of your answer is worth FOUR TIMES as much as the answer you put in the box.

Consider $\frac{d\vec{x}}{dt} = \begin{bmatrix} \alpha & 4 \\ 1 & 1 \end{bmatrix} \vec{x}$ where $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and α is a real-valued parameter for each of the questions below.

(a) **TRUE or FALSE?** "A Hamiltonian function $H(x, y)$ for the given system of ODEs exists only when $\alpha = -1$."

$\boxed{\text{TRUE}}$

$f = \alpha x + 4y \qquad f_x = \alpha$

$g = x + y \qquad g_y = 1$

$f_x = -g_y \Rightarrow \alpha = -1$

Hamiltonian condition is satisfied when $\alpha = -1$

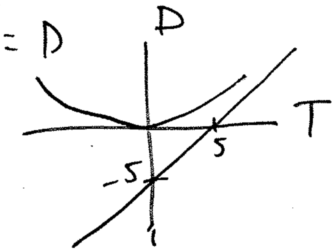
(b) **TRUE or FALSE?** "The curve in the Trace-Determinant plane corresponding to the matrices for all possible values of α is a line through the origin."

$\boxed{\text{FALSE}}$

$T = \alpha + 1 \qquad T - 1 = D + 4 = \alpha$

$D = \alpha - 4 \qquad T - 5 = D$

The curve is a line, but not through the origin



(c) **TRUE or FALSE?** "There is no value of α for which the phase portrait of $\frac{d\vec{x}}{dt} = \begin{bmatrix} \alpha & 4 \\ 1 & 1 \end{bmatrix} \vec{x}$ near the origin will look like the given figure."

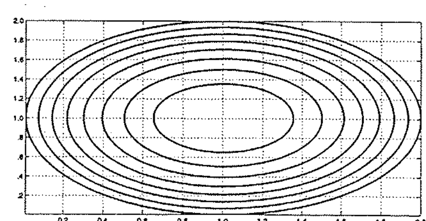
$\boxed{\text{TRUE}}$

$\lambda^2 - (\alpha+1)\lambda + \alpha - 4$

$\lambda = \dfrac{\alpha + 1 \pm \sqrt{(\alpha+1)^2 - 4(\alpha - 4)}}{2}$

$= \dfrac{\alpha + 1 \pm \sqrt{(\alpha-1)^2 + 4^2}}{2}$ always real!

centers only happen along $T = 0, D > 0$



$\beta = \alpha^2 + 1 + 2\alpha - 4\alpha + 16$
$= \alpha^2 - 2\alpha + 1 + 16 = (\alpha - 1)^2 + 4^2$

3. *[30 pts. total]* **Non-Linear Systems, Linearization.**
Consider the following nonlinear system of ordinary differential equations:

$$\frac{dx}{dt} = -\alpha - x + y$$
$$\frac{dy}{dt} = -4x + y + x^2$$

where α is an unknown real parameter.
(a) *[6 points]* Show that the equilibria of the nonlinear system when $\alpha \neq 0$ are $\left(\frac{3+\sqrt{9-4\alpha}}{2}, \frac{3+2\alpha+\sqrt{9-4\alpha}}{2}\right)$ and $\left(\frac{3-\sqrt{9-4\alpha}}{2}, \frac{3+2\alpha-\sqrt{9-4\alpha}}{2}\right)$.

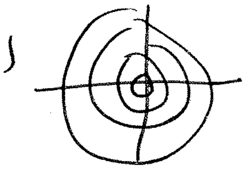
$0 = -\alpha - x + y \Rightarrow y = \alpha + x$
$0 = -4x + y + x^2 \Rightarrow x^2 + \alpha + x - 4x = 0$
$x^2 - 3x + \alpha = 0$
$x = \frac{3 \pm \sqrt{3^2 - 4\alpha}}{2} = \frac{3 \pm \sqrt{9-4\alpha}}{2}$
$y = \alpha + x = \frac{3 + 2\alpha \pm \sqrt{9-4\alpha}}{2}$

(b) *[10 points]* Consider the nonlinear system when $\alpha = 0$. Classify the equilibrium point(s) you found in part (a) by linearizing the nonlinear system for this value of α. What can you say about the appearance of the phase portrait very near to the equilibrium point(s)? (HINT: draw a little sketch!)

when $\alpha = 0$ $\quad x = \frac{3 \pm 3}{2}, \ y = \frac{3 \pm 3}{2}$
$\cancel{x =}$ $(0, 0)$ or $(3, 3)$

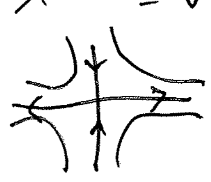
$J = \begin{pmatrix} -1 & 1 \\ 2x-4 & 1 \end{pmatrix}$ $\qquad J(0,0) = \begin{pmatrix} -1 & 1 \\ -4 & 1 \end{pmatrix}$

$\lambda^2 + 0\lambda + 4 = 0 \Rightarrow \lambda = \pm 2i$
center at $(0,0)$



$J(3,3) = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}$

$\lambda^2 + 0\lambda - 3 = 0$
$\lambda = \pm\sqrt{3} \Rightarrow$ saddle at $(3,3)$



You are still considering the system

$$\frac{dx}{dt} = -\alpha - x + y$$
$$\frac{dy}{dt} = -4x + y + x^2$$

(c) *[4 points]* For what bifurcation value $\alpha = \alpha_B$ do the equilibrium points in part (a) coincide?

$$\sqrt{9-4\alpha} = 0 \Rightarrow 4\alpha = 9 \Rightarrow \alpha = \frac{9}{4}$$

Will only have one point when $\sqrt{9-4\alpha} = 0$

(d) *[10 points]* For this particular value $\alpha = \alpha_B$ found in part (c) find the location of the equilibrium point and use the linearization technique again to classify this equilibrium by linearizing the nonlinear system. What can you say about the appearance of the phase portrait very near to this equilibrium point? (HINT: draw a little sketch!)

$$\alpha = \frac{9}{4}$$

$$x = \frac{3}{2}, \quad y = \frac{3 + 2(9/4)}{2}$$

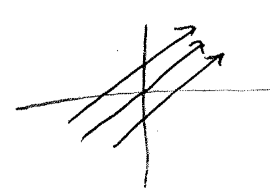
$$x = \frac{3}{2}, \quad y = \frac{15}{4}$$

$$J\left(\frac{3}{2}, \frac{15}{4}\right) = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\lambda^2 - 0\lambda + 0 = 0 \Rightarrow \lambda = 0, 0$$

repeated zero eigenvalues!

Solutions will look like lines parallel to the eigenvector



BONUS *[10 pts. total]* **The Matrix Exponential.**

Another way of writing the solution to an n-dimensional linear system of first-order ODEs $\frac{d\vec{x}}{dt} = A\vec{x}$ is $\vec{x}(t) = e^{At}\vec{c}$ where $\vec{c}$ is a $n \times 1$ vector of unknown constants. The object e^{At} is known as the **matrix exponential** and is a $n \times n$ matrix.

Consider again $\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \vec{x}$ from Question 1. Use the matrix exponential formula above to reproduce the general solution to this ODE you wrote down in **1(c)**. HINT: you may be interested to know that the inverse of $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is $A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

RECALL If A is a diagonalizable matrix, then $A^k = S\Lambda^k S^{-1}$ where k is an integer, S is a matrix of eignevectors of A and Λ is a diagonal matrix of corresponding eigenvalues of A.

$$\vec{x} = e^{At}\vec{c} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix} \frac{1}{3}\vec{c}$$

$$= \frac{1}{3} \begin{pmatrix} e^{2t} & e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & -e^{-t} & -e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} e^{2t}+2e^{-t} & e^{2t}-e^{-t} & e^{2t}-e^{-t} \\ e^{2t}-e^{-t} & e^{2t}+2e^{-t} & e^{2t}-e^{-t} \\ e^{2t}+2e^{-t}+e^{-t} & e^{2t}+e^{-t}-2e^{-t} & e^{2t}+e^{-t}+e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} c_1(e^{2t}+2e^{-t}) + c_2(e^{2t}-e^{-t}) + c_3(e^{2t}-e^{-t}) \\ c_1(e^{2t}-e^{-t}) + c_2(e^{2t}+2e^{-t}) + c_3(e^{2t}-e^{-t}) \\ c_1(e^{2t}+1e^{-t}) + c_2(e^{2t}-1e^{-t}) + c_3(e^{2t}+2e^{-t}) \end{pmatrix}$$

$$= \frac{1}{3}\left(e^{2t}\begin{pmatrix} c_1+c_2+c_3 \\ c_1+c_2+c_3 \\ c_1+c_2+c_3 \end{pmatrix} + e^{-t}\begin{pmatrix} 2c_1-c_2-c_3 \\ c_1+2c_2-c_3 \\ c_1-c_2-2c_3 \end{pmatrix} \cancel{+e^{-t}} \; 2 \right)$$

$$Ae^{2t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + e^{-t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + e^{-t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$