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# Differential Equations

Math 341 Fall 2009  
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MWF 2:30-3:25pm Fowler 110  
<http://faculty.oxy.edu/ron/math/341/09/>

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## Worksheet 23: Monday November 9

**TITLE** Hamiltonian Systems

**CURRENT READING** Blanchard, 5.2 & 5.3

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### Homework Assignments due Friday November 13

Section 5.1: 3, 4, 5, 18, 21.

Section 5.3: 2, 12, 13, 14, 17, 18.

Chapter 5 Review: 3, 4, 5, 6, 7, 8, 11, 12, 25, 27, 28.

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### SUMMARY

We shall continue our analysis of non-linear systems by introducing the concept of a Hamiltonian function.

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Consider the following nonlinear planar system of ODEs

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= x - x^2\end{aligned}$$

### **Exercise**

Show that the function  $H(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3$  has the property that  $\frac{dH}{dt} = 0$  if  $x$  and  $y$  simultaneously satisfy the given system of ODEs. (HINT: Use the Differentiation Chain Rule!)

## 1. The Hamiltonian

### **DEFINITION: Hamiltonian function**

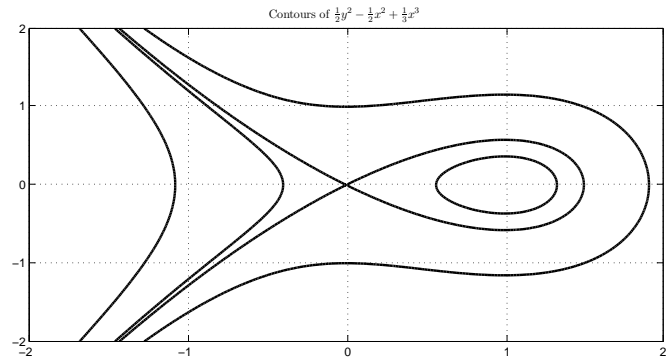
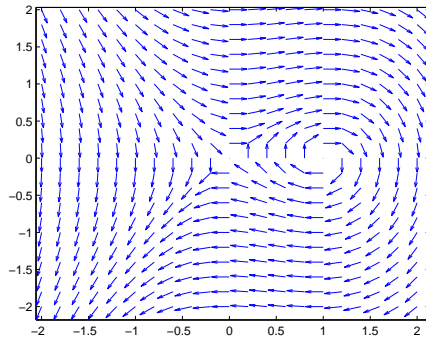
A real-valued function  $H(x, y)$  is considered to be a conserved quantity for a system of ordinary differential equations if it is constant along ALL solution curves of the system. In other words, IF  $(x(t), y(t))$  is a solution of the system then  $H(x(t), y(t))$  is constant for all time which also implies that  $\frac{d}{dt}H(x(t), y(t)) = 0$ . The function  $H(x, y)$  is known as the Hamiltonian function (or Hamiltonian) of the system of ODEs.

## 2. The Hamiltonian Level Curves and The Phase Portrait

### RECALL

The **level curves** or **contours** of the function  $H(x, y)$  are the set of points in the plane which satisfy the equation  $H(x, y) = k$  for certain real values  $k$ .

Let's compare the level curves of  $H(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3$  with the direction field of the system  $\dot{x} = y$ ;  $\dot{y} = x - x^2$ . What do you notice?



## 3. Hamiltonian System

### DEFINITION: Hamiltonian System

A system of differential equations is called a **Hamiltonian system** if there exists a real-valued function  $H(x, y)$  such that

$$\begin{aligned}\frac{dx}{dt} &= \frac{\partial H}{\partial y} \\ \frac{dy}{dt} &= -\frac{\partial H}{\partial x}\end{aligned}$$

for all  $x$  and  $y$ . The function  $H$  is called the Hamiltonian function for the system.

### EXAMPLE

The Hamiltonian often has a physical meaning for the system of ODEs that is modelling a particular real-world situation, since it represents a quantity that is being conserved over time. For example, consider the system of ODEs that represents the **undamped** harmonic oscillator  $y'' + qy = 0$ :

$$\begin{aligned}\frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -qy\end{aligned}$$

Let's show that the Hamiltonian for this system is  $H(y, v) = \frac{1}{2}v^2 + \frac{q}{2}y^2$  which represents the total energy of the oscillator.

#### 4. Obtaining Hamiltonians For Systems

In general the planar nonlinear system of first order DEs looks like

$$\begin{aligned}\frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= g(x, y)\end{aligned}$$

In order to find  $H(x, y)$  we need to solve the following equations

$$\begin{aligned}f(x, y) &= \frac{\partial H}{\partial y} \\ g(x, y) &= -\frac{\partial H}{\partial x}\end{aligned}$$

Does a Hamiltonian exist for this system? Well, if it does (and  $H$  has continuous second partial derivatives) then  $\frac{\partial^2 H}{\partial x \partial y} = \frac{\partial^2 H}{\partial y \partial x}$  which would mean that

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} H_y = \frac{\partial}{\partial y} H_x = -\frac{\partial g}{\partial y}$$

So in order to check whether a given system of ODEs has a Hamiltonian or not all one needs to do is check whether

$$\frac{\partial f}{\partial x} = -\frac{\partial g}{\partial y}$$

#### **Exercise**

Is this a Hamiltonian System? If so, find the Hamiltonian function.

$$\begin{aligned}\frac{dx}{dt} &= x + y^2 \\ \frac{dy}{dt} &= y^2 - x\end{aligned}$$

#### **EXAMPLE**

Is this a Hamiltonian System? If so, find the Hamiltonian function.

$$\begin{aligned}\frac{dx}{dt} &= -x \sin(y) + 2y \\ \frac{dy}{dt} &= -\cos(y)\end{aligned}$$

## 5. Equilibria of Hamiltonian Systems

**Hamiltonian Systems Can Never Have Sources or Sinks As Equilibria.** How could we prove that statement?

Consider

$$\begin{aligned}\frac{dx}{dt} &= \frac{\partial H}{\partial y} \\ \frac{dy}{dt} &= -\frac{\partial H}{\partial x}\end{aligned}$$

at the point  $(x_0, y_0)$  which is the equilibrium point. Let's use the Linearization Technique!

The Jacobian of the linearized version of the Hamiltonian System at  $(x_0, y_0)$  will be

What can we say about its eigenvalues?

What does that allow us to conclude?