Differential Equations

Worksheet 18: **Wednesday October 28**

TITLE Linear Systems with Complex Eigenvalues
CURRENT READING Blanchard, 3.4

Homework Assignments due Friday October 30
Section 3.1: 7, 8, 14, 15.
Section 3.2: 6, 11, 15, 16, 18.
Section 3.3: 3, 4, 7, 8.
Section 3.4: 1, 2, 3, 4.

SUMMARY
We'll continue to explore the various scenarios that occur with linear systems of ODEs. This
time dealing with those that possess complex (or imaginary) eigenvalues.

1. **Two Complex Eigenvalues**

Given a system of linear ODEs with associated matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the characteristic
polynomial is $p(\lambda) = (a-\lambda)(d-\lambda) - bc = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = A\lambda^2 + B\lambda + C = 0$ where
$A = 1$, $B = -(a+d)$ and $C = ad - bc$.

Using the quadratic formula to solve the characteristic quadratic polynomial equation $A\lambda^2 + B\lambda + C = 0$ produces $\lambda = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$.

Previously we showed that the conditions on A, B and C for two real eigenvalues was
$B^2 > 4AC$ which is equivalent to $(a-d)^2 > -4bc$.

Clearly, the condition $B^2 < 4AC$ which is equivalent to $(a-d)^2 < -4bc$ will result in "imaginary" solutions to the characteristic polynomial, i.e. complex eigenvalues.

Also (for completeness), when $(a-d)^2 = -4bc$ there will be only one solution to the quadratic
equation, i.e. repeated eigenvalues equal to $\lambda = \dfrac{(a+d)}{2}$.

2. **Reviewing Complex Arithmetic**

Recall that the $i^2 = -1 \Leftrightarrow i = \sqrt{-1}$ and a complex number z has the form $z = a + bi$ where
a and b are real numbers. We refer to the real part of z as $\mathrm{Re}z$ and is equal to a and the
complex part of z as $\mathrm{Im}z$ and is equal to b.

De Moivre's Formula
One of the most fun formulas in mathematics is DeMoivre's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

When $\theta = \pi$ one gets $e^{i\pi} = -1$ which means that $e^{i\pi} + 1 = 0$ which has all of these important
numbers in it: 0, 1, π, i and e. Some call it the greatest equation, ever!

Complex Conjugates

Complex numbers often show up in pairs, called complex conjugates. A complex conjugate of the number $z = a + bi$ is $z^* = a - bi$. For example, $\sqrt{-4} = 0 \pm 2i$. These numbers $-2i$ and $2i$ are complex conjugates of each other. Note that the product of two complex conjugates is a completely real number.

GROUPWORK

Let's practice some complex arithmetic by simpliflying some expressions into the form $a + bi$.

1. $(1 + 2i)(1 - 2i)$

2. $\dfrac{4}{i}$

3. $e^{-i\pi/2}$

4. $(-1 + 2i)(2 - 3i)$

RECALL

The general solution to $\dfrac{d\vec{x}}{dt} = A\vec{x}$ is $\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$ where $A\vec{v}_1 = \lambda_1 \vec{v}_1$ and $A\vec{v}_2 = \lambda_2 \vec{v}_2$, i.e $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors corresponding to eigenvalues λ_1 and λ_2.

What happens if the eigenvalues are complex? Let's find out!

EXAMPLE

What's the general solution to $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \vec{x}$. Check your answer.

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3. Types Of Solutions With Complex Eigenvalues

Given that one has complex eigenvalues of the form $\alpha \pm i\beta$ there are three possibilities for what the phase portrait behavior of the solution will look like. All the solutions will rotate around the origins with a period of $\dfrac{2\pi}{\beta}$.

CASE 1: $\alpha < 0$ All solutions spiral inwards towards the origin, which is classified as a **stable spiral sink**.

CASE 2: $\alpha > 0$ All solutions spiral outwards from the origin, which is classified as an **unstable spiral source**.

CASE 3: $\alpha = 0$ All solutions follow elliptical or circular paths rotating around the origin, which is classified as an **unstable center**.

GROUPWORK

Come up with your own examples of 2x2 matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ that have complex eigenvalues and the corresponding ODE for each of the three cases named above. Use `HPGSystemsSolver` or `pplane` to help you sketch the phase portrait for each case on the given axes.
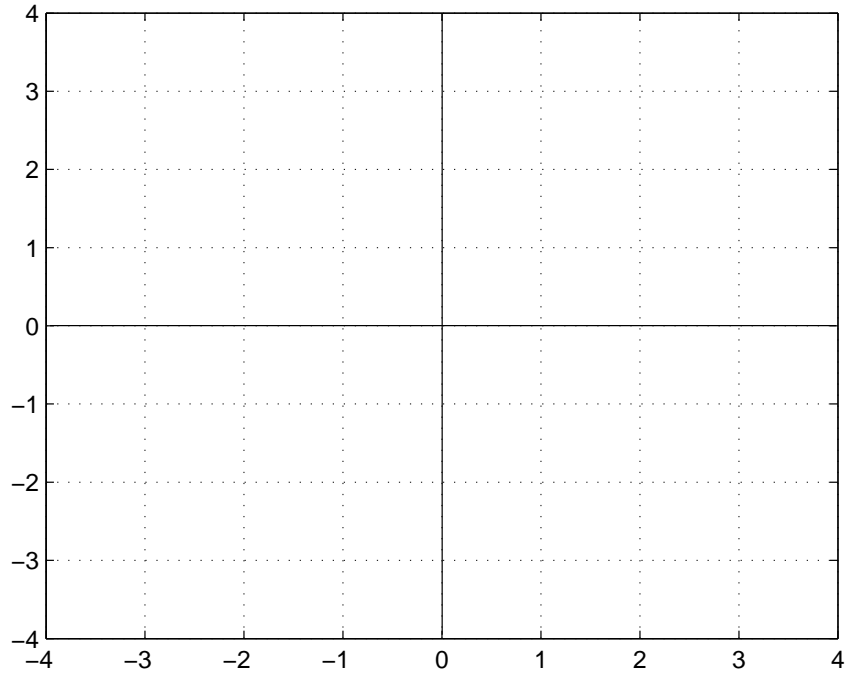
Recall, $\lambda = \alpha \pm \beta i = \dfrac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}$. What values of a,b,c and d will produce α equalling zero? α negative? α positive? Or you could just explore with parameters until you obtain the phase portrait you are looking for in each case.
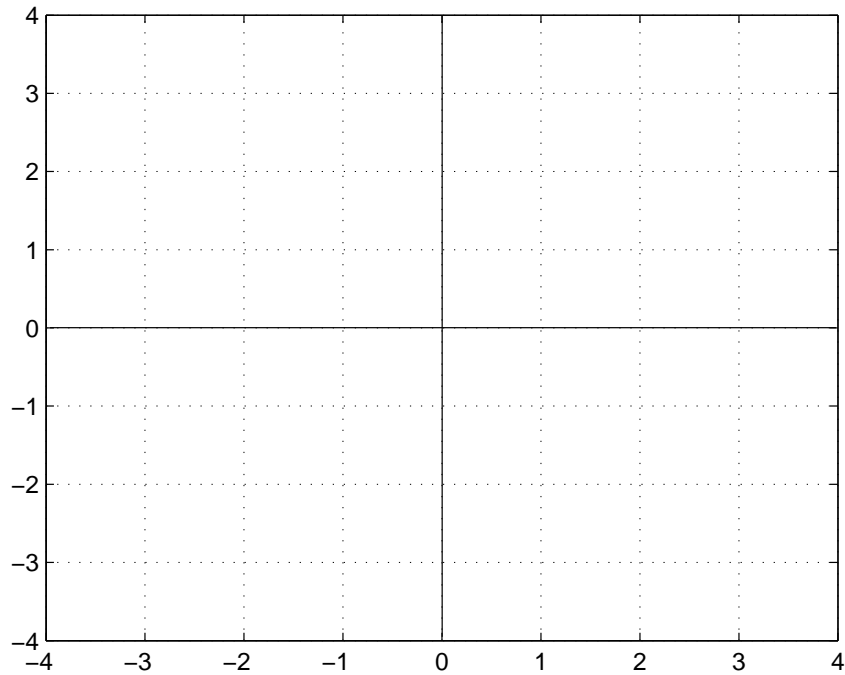
4. CASE 1: spiral sink
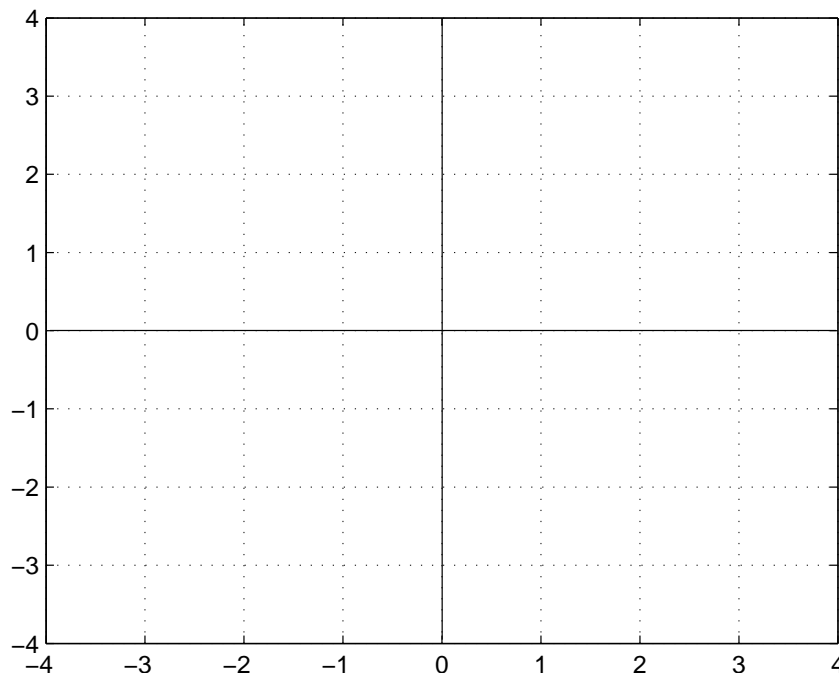
Corresponding ODE: $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} \end{bmatrix} \vec{x}$

5. **CASE 2: spiral source**

Corresponding ODE: $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} & \end{bmatrix} \vec{x}$

6. **CASE 3: unstable center**

Corresponding ODE: $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} & \end{bmatrix} \vec{x}$