Differential Equations

Math 341 Fall 2009 MWF 2:30-3:25pm Fowler 110

 `http://faculty.oxy.edu/ron/math/341/09/`

Worksheet 16: **Friday October 16**

TITLE Straight Line Solutions
CURRENT READING Blanchard, 3.2

Homework Assignments due Friday October 30
Section 3.1: 7, 8, 14, 15.
Section 3.2: 6, 11, 15, 16, 18.
Section 3.3: 3, 4, 7, 8.
Section 3.4: 1, 2, 3, 4.

SUMMARY
Eigenvalues and eigenvectors return and are important in the case where Linear Systems of ODEs have solutions that look like straight lines.

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1. The Significance of Eigenvectors and Eigenvalues

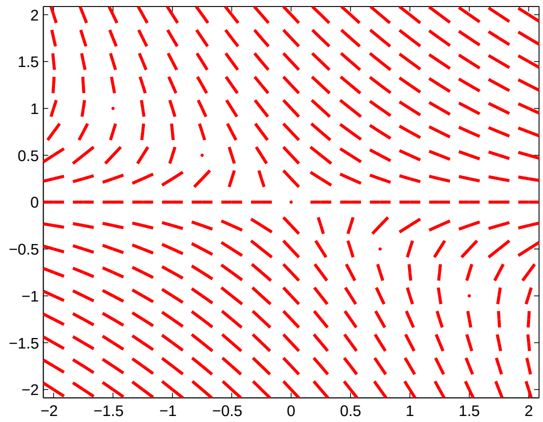
Recall the solutions $\vec{x}_1(t) = \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}$ and $\vec{x}_2(t) = \begin{bmatrix} -e^{-4t} \\ 2e^{-4t} \end{bmatrix}$ to the ODE $\frac{d\vec{x}}{dt} = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix} \vec{x}$ from *Worksheet #15*.

Notice that $\vec{x}_1(t) = \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t}$ and $\vec{x}_2(t) = \begin{bmatrix} -e^{-4t} \\ 2e^{-4t} \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-4t}$.

Question
Do you notice anything interesting about the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$? Any relationship to the matrix $A = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix}$? What happens if you multiply each vector by A?

Answer
The vectors in question are ____.

Consider the slope field for the ODE $\frac{d\vec{x}}{dt} = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix} \vec{x}$:



It turns out that the general solution to $\frac{d\vec{x}}{dt} = \begin{bmatrix} 2 & 3 \\ 0 & -4 \end{bmatrix} \vec{x}$ can be written as
$\vec{x} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-4t}$.

Exercise

On the above slope field, draw in the solutions $\vec{x}_1(t)$ and $\vec{x}_2(t)$. What happens as $t \to \infty$? What about as $t \to -\infty$?

Draw in the nullclines also. Is there any general relationship between the straight-line solutions and the nullclines?

EXAMPLE

Consider the system $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \vec{x}$. Find the eigenvalues λ and eigenvectors $\vec{v}$ of $\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}$.

Show that the general solution can be written as $\vec{x} = c_1\vec{v}_1 e^{\lambda_1 t} + c_1\vec{v}_2 e^{\lambda_2 t}$ and confirm that it is actually a solution of $\vec{x}' = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \vec{x}$.

2. General Solution To Homogeneous Linear Systems

THEOREM

The general solution $\vec{x}(t)$ on the interval $(-\infty, \infty)$ to a homogeneous system of linear DEs $\frac{d\vec{x}(t)}{dt} = A(t)\vec{x}(t)$ can be written as $\vec{x} = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t} + c_3\vec{v}_3 e^{\lambda_3 t} + \ldots + c_n\vec{v}_n e^{\lambda_n t}$ where λ_1, λ_2, λ_3, …, λ_n and $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, …, $\vec{v}_n$ are the eigenvalues and corresponding eigenvectors of the matrix A.

3. Phase Portraits With Straight Line Solutions

Exercise

Solve $\dfrac{dx}{dt} = 2x + 2y, \quad \dfrac{dy}{dt} = x + 3y.$

GroupWork

Use `HPGSystemSolver` to sketch the phase portrait of the linear system $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \vec{x}$ you solved above, in the space below.