Math 341 Fall 2008 **SHOW ALL YOUR WORK** Quiz **Five**

1. Consider the linear system of ordinary differential equations with a parameter a

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 2 & a \\ 2 & 0 \end{bmatrix} \vec{x} \text{ where } \vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

Let's describe how the phase portrait changes as a varies from $-\infty$ to $+\infty$.

(a) *3 points.* Compute all the eigenvalues and eigenvectors in order to sketch the phase portrait when $a = -3/2$. Describe the stationary point at the origin.

$\lambda^2 - 2\lambda + 3 = 0$

$\lambda = \dfrac{2 \pm \sqrt{(-2)^2 - 4\cdot1\cdot3}}{2}$

$\lambda = \dfrac{2 \pm \sqrt{-8}}{2} = \dfrac{2 \pm i2\sqrt{2}}{2} = 1 \pm i\sqrt{2}$

origin is a SPIRAL SOURCE since $Re(\lambda) > 0$.

$E_{1+i\sqrt{2}} = span\left\{\begin{pmatrix} 1+i\sqrt{2} \\ 2 \end{pmatrix}\right\}$ $\left(\begin{array}{cc|c} 2 & -1-i\sqrt{2} & 0 \\ 0 & 0 & 0 \end{array}\right)$

$E_{1-i\sqrt{2}}: span \begin{pmatrix} 1-i\sqrt{2} \\ 2 \end{pmatrix}$ $E_{1+i\sqrt{2}} = null \begin{pmatrix} 2-(1+i\sqrt{2}) & -3/2 \\ 2 & -(1+i\sqrt{2}) \end{pmatrix}$

$\left(\begin{array}{cc|c} 1-i\sqrt{2} & -3/2 & 0 \\ 2 & -1-i\sqrt{2} & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 3 & -\frac{3}{2}(1+i\sqrt{2}) & 0 \\ 2 & -1-i\sqrt{2} & 0 \end{array}\right)$

(b) *3 points.* Compute all the eigenvalues and eigenvectors in order to sketch the phase portrait when $a = 1$. Describe the stationary point at the origin.

origin is a SADDLE.

$\lambda^2 - 2\lambda - 2 = 0$

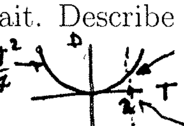
$\lambda = \dfrac{2 \pm \sqrt{2^2 - 4(-2)}}{2} = \dfrac{2 \pm \sqrt{12}}{2} = 1 \pm \sqrt{3} = \underset{>0}{1+\sqrt{3}}, \underset{<0}{1-\sqrt{3}}$

$E_{1+\sqrt{3}} = \left(\begin{array}{cc|c} 1-\sqrt{3} & 1 & 0 \\ 2 & -1-\sqrt{3} & 0 \end{array}\right) \to \left(\begin{array}{cc|c} -2 & 1+\sqrt{3} & 0 \\ 2 & -1-\sqrt{3} & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & \frac{-1-\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 \end{array}\right) = span \begin{pmatrix} \frac{1+\sqrt{3}}{2} \\ 1 \end{pmatrix}$

$E_{1-\sqrt{3}} = \left(\begin{array}{cc|c} 1+\sqrt{3} & 1 & 0 \\ 2 & -1+\sqrt{3} & 0 \end{array}\right) \to \left(\begin{array}{cc|c} -2 & 1-\sqrt{3} & 0 \\ 2 & -1+\sqrt{3} & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -\frac{1+\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 \end{array}\right) = span \begin{pmatrix} \frac{-1+\sqrt{3}}{2} \\ -1 \end{pmatrix}$

(c) *4 points.* For what value of a does the system change its nature (i.e. bifurcate)? For this value of a, compute the eigenvalues and eigenvectors in order to sketch the phase portrait. Describe the stationary point at the origin. $\lambda^2 - 2\lambda + (-2a)$

$T = 2$ $D = -2a$ $D = \frac{T^2}{4}$



$a = -\frac{1}{2}$ +1 BONUS

$a = 0$ another bifurcation

$\lambda = \dfrac{2 \pm \sqrt{(-2)^2 - 4\cdot1\cdot(-2a)}}{2}$

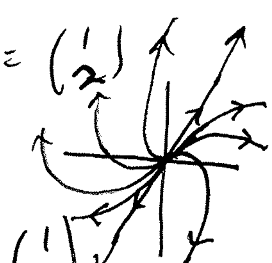
$\lambda = \dfrac{2 \pm \sqrt{4+8a}}{2} = 1 \pm \sqrt{1+2a}$ $\Rightarrow$ bifurcation occurs when $1+2a=0 \Rightarrow a = -1/2$

When $a = -1/2$, $\lambda = 1, 1$

The origin is an UNSTABLE IMPROPER NODE

Need E_1 $E_1 = span\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\} \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\left(\begin{array}{cc|c} 1 & -1/2 & 0 \\ 2 & -1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$



Solve $(A - \lambda I)\vec{w} = \vec{v}$ $\vec{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\left(\begin{array}{cc|c} 2-1 & -1/2 & 1 \\ 2 & 0-1 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1/2 & 1 \\ 2 & -1 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 2 & -1 & 2 \\ 0 & 0 & 0 \end{array}\right)$

$\vec{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^t + c_2 \left[t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] e^t$