Differential Equations

Math 341 Fall 2008 | Fowler 307 MWF 2:30pm - 3:25pm

 | http://faculty.oxy.edu/ron/math/341/08/

Week 13

Monday November 17 : *Worksheet 31*

Laplace Transform and the Delta Function. We shall be introduced to one of the strangest function in all of mathematics, the Dirac Delta Function, $\delta(t)$ and how it can be used to solve problems that have an impulse forcing using Laplace Transforms.

Reading:

Blanchard, Section 6.4

Homework #30:

Blanchard, Section 6.4: 1, 2, 5, 7, 8.

Quiz:

Reading Quiz #5.

Wednesday November 19 : *Worksheet 32*

Laplace Transforms and Convolutions. We shall discuss the equivalent of the product rule for Laplace Transforms and be introduced to the concept of the convolution of two functions.

Reading:

Blanchard, 6.5

Homework #31:

Blanchard, Section 6.5: 2, 5, 6, 9.

Friday November 21 : *Worksheet 33*

Revisiting Euler's Method. We shall re-visit Euler's Method and explore the error term.

Reading:

Blanchard, Section 7.1 and 7.2

Homework #32:

Blanchard, Section 7.1: 3, 7, 10.

Blanchard, Section 7.2: 1, 7, 8, 13.

Quiz:

Take-Home Quiz #7.

35. **(a)** Consider

$$\mathcal{L}[f] = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt.$$

We can calculate dF/ds by differentiating under the integral sign. That is,

$$\begin{aligned}\frac{dF}{ds} &= \int_0^\infty \frac{\partial}{\partial s}\left(f(t)\, e^{-st}\right) dt \\ &= \int_0^\infty f(t)(-t)e^{-st}\, dt \\ &= -\mathcal{L}[tf(t)].\end{aligned}$$

(b) If we apply this result to

$$\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2+\omega^2} = \omega(s^2+\omega^2)^{-1},$$

we obtain

$$\begin{aligned}\mathcal{L}[t \sin \omega t] &= -\omega(-1)(s^2+\omega^2)^{-2}(2s) \\ &= \frac{2\omega s}{(s^2+\omega^2)^2}.\end{aligned}$$

Compare this result with the result of Exercise 6.

XERCISES FOR SECTION 6.4

1. This is the $\frac{0}{0}$ case of L'Hôpital's Rule. Differentiating numerator and denominator with respect to Δt, we obtain

$$\frac{se^{s\Delta t} - (-s)e^{-s\Delta t}}{2},$$

which simplifies to

$$\frac{s(e^{s\Delta t} + e^{-s\Delta t})}{2}.$$

Since both $e^{s\Delta t}$ and $e^{-s\Delta t}$ tend to 1 as $\Delta t \to 0$, the desired limit is s.

2. Taking Laplace transforms of both sides and applying the rules yields

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 3\mathcal{L}[y] = 5\mathcal{L}[\delta_2].$$

Simplifying, using the initial conditions, and the fact that $\mathcal{L}[\delta_2] = e^{-2s}$, we get

$$(s^2+3)\mathcal{L}[y] = 5e^{-2s}.$$

Hence,

$$\mathcal{L}[y] = 5\frac{e^{-2s}}{s^2+3}.$$

This can be written as

$$\mathcal{L}[y] = \frac{5}{\sqrt{3}} e^{-2s} \frac{\sqrt{3}}{s^2+3},$$

which yields

$$y(t) = \frac{5}{\sqrt{3}} u_2(t) \sin\left(\sqrt{3}(t-2)\right).$$

3. Applying the Laplace transform to both sides, using the rules, and the fact that $\mathcal{L}[\delta_3] = e^{-3s}$, we get

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 2s\mathcal{L}[y] - 2y(0) + 5\mathcal{L}[y] = e^{-3s}.$$

Substituting the given initial conditions, we have

$$\mathcal{L}[y] = \frac{s+3}{s^2+2s+5} + \frac{e^{-3s}}{s^2+2s+5}.$$

Using the fact that $s^2+2s+5 = (s+1)^2+4$, we obtain

$$\mathcal{L}[y] = \frac{s+1}{(s+1)^2+4} + \frac{2}{(s+1)^2+4} + \frac{1}{2}e^{-3s}\frac{2}{(s+1)^2+4}.$$

Therefore,

$$y(t) = e^{-t}\cos 2t + e^{-t}\sin 2t + \frac{1}{2}u_3(t)e^{-(t-3)}\sin(2(t-3)).$$

4. Taking the Laplace transform of both sides, using the rules, and the fact that $\mathcal{L}[\delta_2] = e^{-2s}$, we get

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 2s\mathcal{L}[y] - 2y(0) + 2\mathcal{L}[y] = -2e^{-2s}.$$

Substituting the given initial conditions, we obtain

$$\mathcal{L}[y] = \frac{2s+4}{s^2+2s+2} - \frac{2e^{-2s}}{s^2+2s+2}.$$

Using $s^2+2s+2 = (s+1)^2+1$ in the denominator gives us

$$\mathcal{L}[y] = 2\frac{s+1}{(s+1)^2+1} + 2\frac{1}{(s+1)^2+1} - 2e^{-2s}\frac{1}{(s+1)^2+1}.$$

Taking the inverse Laplace transform, we have

$$y(t) = 2e^{-t}\cos t + 2e^{-t}\sin t - 2u_2(t)e^{-(t-2)}\sin(t-2).$$

5. Applying Laplace transform to both sides, using the rules, and the fact that $\mathcal{L}[\delta_a] = e^{-as}$, we get

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 2s\mathcal{L}[y] - 2y(0) + 3\mathcal{L}[y] = e^{-s} - 3e^{-4s}.$$

Substituting the initial conditions gives us

$$\mathcal{L}[y] = \frac{e^{-s}}{s^2+2s+3} - \frac{3e^{-4s}}{s^2+2s+3}.$$

Now, using that $s^2+2s+3 = (s+1)^2+2$, we have

$$\mathcal{L}[y] = \frac{1}{\sqrt{2}}e^{-s}\frac{\sqrt{2}}{(s+1)^2+2} + \frac{3}{\sqrt{2}}e^{-4s}\frac{\sqrt{2}}{(s+1)^2+2}.$$

So,

$$y(t) = \frac{1}{\sqrt{2}}u_1(t)e^{-(t-1)}\sin(\sqrt{2}(t-1)) - \frac{3}{\sqrt{2}}u_4(t)e^{-(t-4)}\sin(\sqrt{2}(t-4)).$$

6. **(a)** The characteristic polynomial of the unforced oscillator is $\lambda^2+2\lambda+3$, and the eigenvalues are $\lambda = -1 \pm \sqrt{2}\,i$. Hence, the natural period is $\sqrt{2}\,\pi$ and the damping causes the solutions of the unforced equation to tend to zero like e^{-t}. At $t = 4$, the system is given a jolt, so the solution rises. After $t = 4$, the equation is unforced, so the solution again tends to zero as e^{-t}.

(b) Taking Laplace transforms of both sides of the equation, we have

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 2s\mathcal{L}[y] - 2y(0) + 3\mathcal{L}[y] = \mathcal{L}[\delta_4].$$

Plugging in the initial conditions and solving for $\mathcal{L}[y]$ gives us

$$\mathcal{L}[y] = \frac{s+2}{s^2+2s+3} + \frac{e^{-4s}}{s^2+2s+3}.$$

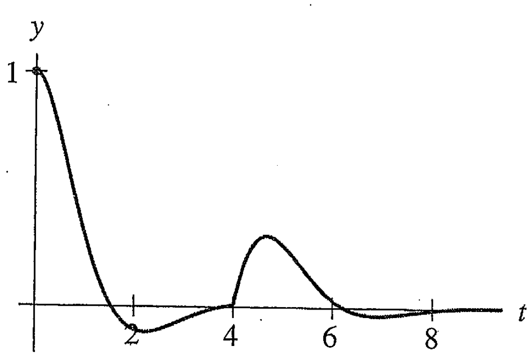
If we complete the square for the polynomial s^2+2s+3, we get $s^2+2s+3 = (s+1)^2+2$, so

$$\mathcal{L}[y] = \frac{s+1}{(s+1)^2+2} + \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+2} + \frac{1}{\sqrt{2}}e^{-4s}\frac{\sqrt{2}}{(s+1)^2+2}.$$

Therefore,

$$y(t) = e^{-t}\cos\sqrt{2}\,t + \frac{1}{\sqrt{2}}e^{-t}\sin\sqrt{2}\,t + \frac{1}{\sqrt{2}}u_4(t)e^{-(t-4)}\sin(\sqrt{2}(t-4)).$$

(c)



Note that the solution goes through about 3/4 of a natural period before the application of the delta function. The delta function forcing causes the second maximum of the solution to be much higher than it would have been without the forcing, but the long term effect is small because the damping is fairly large.

7. **(a)** From the table

$$\mathcal{L}[\delta_a] = e^{-as}$$

$$s\mathcal{L}[u_a] - u_a(0) = s\frac{e^{-as}}{s} - 0 = e^{-as}.$$

(b) The formula for the Laplace transform of a derivative is

$$\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$$

and this is exactly the relationship between the Laplace transforms of $u_a(t)$ and $\delta_a(t)$. Hence, it is tempting to think of the Dirac delta function as the derivative of the Heaviside function.

(c) We can think of the Heaviside function $u_a(t)$ as a limit of piecewise linear functions equal to zero for t less than $a-\Delta t$, equal to one for t greater than $a+\Delta t$ and a straight line for t between $a-\Delta t$ and $a+\Delta t$. The derivative of this function is precisely the function $g_{\Delta t}$ used to define the Dirac delta function. This is still just an informal relationship until we specify in what sense we are taking the limit.

8. Actually, this exercise is a little more complicated than it seems at first. We can think of g as a periodic function with period a and apply Exercise 16 in Section 6.2, but to do so, we must decide how to integrate $\delta_a(t)$ over the interval $0 \le t \le a$. In other words, is the impulse inside or outside the interval?

To avoid this issue, we consider the function

$$f(t) = \sum_{n=0}^{\infty} \delta_{na+a/2}(t).$$

We can apply the periodicity formula from Exercise 16 in Section 6.2 to this function to get

$$\mathcal{L}[f] = \frac{1}{1-e^{-as}}\int_0^a f(t)\,e^{-st}\,dt = \frac{1}{1-e^{-as}}\int_0^a \delta_{a/2}(t)\,e^{-st}\,dt,$$

because $\delta_{na+a/2}(t) = 0$ for all $n > 0$ on the interval $[0, a]$. Moreover,

$$\int_0^a \delta_{a/2}(t)\,e^{-st}\,dt = \mathcal{L}[\delta_{a/2}]$$

because $\delta_{a/2}(t) = 0$ for all $t > a/2$. Therefore, we have

$$\mathcal{L}[f] = \frac{e^{-as/2}}{1-e^{-as}}.$$

To obtain $\mathcal{L}[g]$, we use the relation $g(t) = u_{a/2}(t)f(t-a/2)$ to obtain

$$\mathcal{L}[g] = e^{-as/2}\frac{e^{-as/2}}{1-e^{-as}} = \frac{e^{-as}}{1-e^{-as}}.$$

Note that this is the same answer we get if we apply the periodicity formula directly to $g(t)$ assuming that the entire impulse takes place inside the interval $0 \le t \le a$. In other words, if we assume that

$$\int_0^a \delta_a(t)\,e^{-st}dt = e^{-as},$$

then we get

$$\begin{aligned}\mathcal{L}[g] &= \frac{1}{1-e^{-as}} \int_0^a g(t)\, e^{-st}\, dt \\ &= \frac{1}{1-e^{-as}} \int_0^a \delta_a(t)\, e^{-st}\, dt \\ &= \frac{e^{-as}}{1-e^{-as}}.\end{aligned}$$

9. **(a)** To compute the Laplace transform of the infinite sum on the right-hand side of the equation, we can either sum the geometric series that results from the fact that $\mathcal{L}[\delta_n] = e^{-ns}$ or use Exercise 16 in Section 6.2. Either way, we get

$$\mathcal{L}\left[\sum_{n=1}^{\infty} \delta_n(t)\right] = \frac{e^{-s}}{1-e^{-s}} = \frac{1}{e^s - 1}.$$

For our purposes, it is actually better to leave the Laplace transform of the right-hand side as

$$\mathcal{L}\left[\sum_{n=1}^{\infty} \delta_n(t)\right] = \sum_{n=1}^{\infty} e^{-ns}.$$

Since $y(0) = 0$ and $y'(0) = 0$, the transformed equation is

$$s^2 \mathcal{L}[y] + 2\mathcal{L}[y] = \sum_{n=1}^{\infty} e^{-ns},$$

which simplifies to

$$\mathcal{L}[y] = \frac{1}{s^2+2} \sum_{n=1}^{\infty} e^{-ns} = \sum_{n=1}^{\infty} \frac{e^{-ns}}{s^2+2}.$$

(b) Since

$$\mathcal{L}^{-1}\left[\frac{e^{-ns}}{s^2+2}\right] = \frac{1}{\sqrt{2}} u_n(t) \sin(\sqrt{2}\,(t-n)),$$

we have

$$y(t) = \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} u_n(t) \sin(\sqrt{2}\,(t-n)).$$

(c) The period of the forcing is different from the natural period of the unforced oscillator. Hence, the solution oscillates but not periodically.

10. **(a)** To compute the Laplace transform of the infinite sum on the right-hand side of the equation, we can either sum the geometric series or use Exercise 16 in Section 6.2 (see Exercise 9 as well). We get

$$\mathcal{L}\left[\sum_{n=1}^{\infty} \delta_{2n\pi}(t)\right] = \frac{e^{-2\pi}}{1-e^{-2\pi}}.$$

Section 6.5

Checking the convolution property ($\mathcal{L}[f * g] = \mathcal{L}[f] \cdot \mathcal{L}[g]$) for Laplace transforms, we have

$$\mathcal{L}[f] = \frac{1}{s}, \quad \mathcal{L}[g] = \frac{1}{s+1},$$

and

$$\begin{aligned}
\mathcal{L}[f * g] &= \frac{1}{s} - \frac{1}{s+1} \\
&= \frac{s+1-s}{s(s+1)} \\
&= \frac{1}{s(s+1)}.
\end{aligned}$$

So, $\mathcal{L}[f] \cdot \mathcal{L}[g] = \mathcal{L}[f * g]$.

2. Using the definition of the convolution with f and g, we see that

$$\begin{aligned}
(f * g)(t) &= \int_0^t e^{-a(t-u)} e^{-bu}\, du \\
&= \int_0^t e^{-at} e^{(a-b)u}\, du \\
&= -e^{-at} \frac{e^{(a-b)u}}{a-b}\bigg|_0^t \\
&= e^{-at}\left(\frac{e^{(a-b)t}}{a-b} - \frac{1}{a-b}\right) \\
&= \frac{e^{-bt}}{a-b} - \frac{e^{-at}}{a-b}.
\end{aligned}$$

Checking the convolution property ($\mathcal{L}[f * g] = \mathcal{L}[f] \cdot \mathcal{L}[g]$) for Laplace transforms, we have

$$\mathcal{L}[f] = \frac{1}{s+a}, \quad \mathcal{L}[g] = \frac{1}{s+b},$$

and

$$\begin{aligned}
\mathcal{L}[f * g] &= \frac{1}{(s+b)(a-b)} - \frac{1}{(s+a)(a-b)} \\
&= \frac{s+a}{(s+a)(s+b)(a-b)} - \frac{s+b}{(s+a)(s+b)(a-b)} \\
&= \frac{a-b}{(s+a)(s+b)(a-b)} \\
&= \frac{1}{(s+a)(s+b)}.
\end{aligned}$$

Therefore, $\mathcal{L}[f] \cdot \mathcal{L}[g] = \mathcal{L}[f * g]$.

So

$$\int_0^t u_2(t-v)u_3(v)\,dv = \begin{cases} 0, & \text{if } t < 5, \\ \displaystyle\int_3^{t-2} 1\,dv, & \text{if } t \geq 5. \end{cases}$$

Evaluating the second integral, we get

$$\int_3^{t-2} 1\,dv = t\Big|_3^{t-2} = t - 5.$$

We have a function that is 0 for $t < 5$ and equal to $t - 5$ for $t \geq 5$, so our function is $u_5(t)(t-5)$.

Checking the convolution property ($\mathcal{L}[f * g] = \mathcal{L}[f] \cdot \mathcal{L}[g]$) for Laplace transforms, we have

$$\mathcal{L}[f] = \frac{e^{-2s}}{s}, \quad \mathcal{L}[g] = \frac{e^{-3s}}{s},$$

and

$$\mathcal{L}[f * g] = \frac{e^{-5s}}{s^2}.$$

So, $\mathcal{L}[f] \cdot \mathcal{L}[g] = \mathcal{L}[f * g]$.

5. Using the definition of the convolution with f and g, we see that

$$(f * g)(t) = \int_0^t 3\sin(t-u)\cos(2u)\,du.$$

We will use four trigonometric identities to evaluate this integral:

$$\sin(t-u) = \sin t \cos u - \cos t \sin u$$

$$\sin(mt)\sin(nt) = \tfrac{1}{2}\left[\cos((m-n)t) - \cos((m+n)t)\right]$$

$$\cos(mt)\cos(nt) = \tfrac{1}{2}\left[\cos((m+n)t) + \cos((m-n)t)\right]$$

$$\sin(mt)\cos(nt) = \tfrac{1}{2}\left[\sin((m+n)t) + \sin((m-n)t)\right].$$

So

$$\int_0^t 3\sin(t-u)\cos(2u)\,du$$

$$= \int_0^t \left[3\cos 2u \cos u \sin t - \cos 2u \sin u \cos t\right] du$$

$$= \int_0^t \left[\tfrac{3}{2}(\cos 3u + \cos u)\sin t - \tfrac{3}{2}(\sin 3u - \sin u)\cos t\right] du$$

$$= \sin t \left[\tfrac{1}{2}\sin 3u + \tfrac{3}{2}\sin u\right]_0^t + \cos t \left[\tfrac{1}{2}\cos 3u - \tfrac{3}{2}\cos u\right]_0^t$$

$$= \sin t \left(\tfrac{1}{2}\sin 3t + \tfrac{3}{2}\sin t\right) + \cos t \left(\tfrac{1}{2}\cos 3t - \tfrac{3}{2}\cos t + 1\right)$$

$$
\begin{aligned}
&= \tfrac{1}{2}\sin 3t \sin t + \tfrac{3}{2}\sin^2 t + \tfrac{1}{2}\cos 3t \cos t - \tfrac{3}{2}\cos^2 t + \cos t \\
&= \tfrac{1}{4}(\cos 2t - \cos 4t) + \tfrac{3}{2}\sin^2 t + \tfrac{1}{4}(\cos 4t + \cos 2t) - \tfrac{3}{2}\cos^2 t + \cos t \\
&= \tfrac{1}{2}\cos 2t + \tfrac{3}{2}\left(\sin^2 t - \cos^2 t\right) + \cos t \\
&= \tfrac{1}{2}\cos 2t - \tfrac{3}{2}\cos 2t + \cos t \\
&= \cos t - \cos 2t,
\end{aligned}
$$

which is the same answer obtained in the text using the technique of Laplace transforms.

6. We will use the substitution $v = t - u$, so that $u = t - v$, and $du = -dv$. Also, as u goes from 0 to t, v goes from t to 0, so we have

$$
\begin{aligned}
(f * g)(t) &= \int_0^t f(t-u)g(u)\,du \\
&= -\int_t^0 f(v)g(t-v)\,dv \\
&= \int_0^t f(v)g(t-v)\,dv \\
&= (g * f)(t).
\end{aligned}
$$

7. Taking Laplace transform of both sides of the equation and solving for $\mathcal{L}[\zeta]$ (see page 607), we obtain

$$\mathcal{L}[\zeta] = \frac{1}{s^2 + ps + q}.$$

Hence, if we let

$$z(s) = s^2 + ps + q,$$

we have that $z(0) = 5$ and $z(2) = 17$. Now $z(0) = 5$ implies $q = 5$. Using $z(2) = 17 = 2^2 + 2p + 5$, we see that $p = 4$.

8. Since $\eta(t)$ solves the first equation, we know that

$$\frac{d\eta}{dt} + a\eta = f(t), \quad \eta(0) = 0.$$

Taking the Laplace transform of both sides of the equation, we get

$$s\mathcal{L}[\eta] - \eta(0) + a\mathcal{L}[\eta] = \mathcal{L}[f].$$

Substituting the initial condition and solving for $\mathcal{L}[\eta]$, we have

$$\mathcal{L}[\eta] = \frac{\mathcal{L}[f]}{s+a}.$$

Now, since $\zeta(t)$ solves the second equation, we know that

$$\frac{d\zeta}{dt} + a\zeta = \delta_0.$$

So

$$s\mathcal{L}[\zeta] + a\mathcal{L}[\zeta] = \mathcal{L}[\delta_0],$$

and

$$\mathcal{L}[\zeta] = \frac{1}{s+a}.$$

Hence,

$$\mathcal{L}[\zeta] \cdot \mathcal{L}[f] = \mathcal{L}[\eta].$$

9. **(a)** Since ζ solves the initial-value problem above, we know that

$$\frac{d\zeta^2}{dt^2} + p\frac{d\zeta}{dt} + q\zeta = \delta_0(t), \quad \zeta(0) = \zeta'(0) = 0^-.$$

Taking Laplace transforms of both sides, and substituting initial conditions gives us

$$s^2\mathcal{L}[\zeta] + ps\mathcal{L}[\zeta] + q\mathcal{L}[\zeta] = 1,$$

which yields

$$\mathcal{L}[\zeta] = \frac{1}{s^2 + ps + q}.$$

Now, taking Laplace transforms of both sides of

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0, \quad y(0) = a,\ y'(0) = 0$$

gives us

$$s^2\mathcal{L}[y] - sa + ps\mathcal{L}[y] - pa + q\mathcal{L}[y] = 0.$$

Solving for $\mathcal{L}[y]$ gives

$$\mathcal{L}[y] = \frac{a(s+p)}{s^2 + ps + q},$$

so

$$\mathcal{L}[y] = a(s+p)\mathcal{L}[\zeta].$$

(b) Taking Laplace transforms of both sides of

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0, \quad y(0) = 0,\ y'(0) = b$$

gives us

$$s^2\mathcal{L}[y] - b + ps\mathcal{L}[y] + q\mathcal{L}[y] = 0.$$

Solving for $\mathcal{L}[y]$ gives

$$\mathcal{L}[y] = \frac{b}{s^2 + ps + q},$$

so

$$\mathcal{L}[y] = b\mathcal{L}[\zeta].$$

(c) Taking Laplace transforms of both sides of

$$\frac{d^2y}{dt^2}+p\frac{dy}{dt}+qy=f(t), \quad y(0)=a, y'(0)=b$$

gives us

$$s^2\mathcal{L}[y]-sa-b+ps\mathcal{L}[y]-pa+q\mathcal{L}[y]=\mathcal{L}[f].$$

Solving for $\mathcal{L}[y]$ gives

$$\mathcal{L}[y]=\frac{\mathcal{L}[f]+a(s+p)+b}{s^2+ps+q},$$

so

$$\mathcal{L}[y]=(\mathcal{L}[f]+a(s+p)+b)\,\mathcal{L}[\zeta].$$

10. Since η solves the first initial-value problem, we know that

$$\frac{d\eta^2}{dt^2}+p\frac{d\eta}{dt}+q\eta=u_0(t), \quad \eta(0)=\eta'(0)=0^-.$$

Taking Laplace transforms of both sides and replacing the initial conditions gives us

$$s^2\mathcal{L}[\eta]+ps\mathcal{L}[\eta]+q\mathcal{L}[\eta]=\frac{1}{s}.$$

Solving for $\mathcal{L}[\eta]$ gives

$$\mathcal{L}[\eta]=\frac{1}{s(s^2+ps+q)}.$$

If we take the Laplace transform of both sides of the second initial-value problem, and solve for $\mathcal{L}[y]$, we have

$$\mathcal{L}[y]=\frac{\mathcal{L}[f]}{s^2+ps+q}.$$

Using the convolution property for Laplace Transforms, we get

$$\begin{aligned}\mathcal{L}[y]&=s\left(\mathcal{L}[f]\cdot\mathcal{L}[\eta]\right)\\&=s\left(\mathcal{L}[f*\eta]\right).\end{aligned}$$

Now,

$$(f*\eta)(t)=\int_0^t f(t-u)\eta(u)\,du,$$

so

$$(f*\eta)(0)=\int_0^0 f(t-u)\eta(u)\,du=0.$$

Using the rule that

$$\mathcal{L}\left[\frac{dy}{dt}\right]=s\mathcal{L}[y]-y(0),$$

we have that

$$\begin{aligned}\mathcal{L}\left[\frac{d}{dt}(f*\eta)\right]&=s\mathcal{L}[(f*\eta)]-(f*\eta)(0)\\&=s\mathcal{L}[(f*\eta)]\\&=\mathcal{L}[y].\end{aligned}$$

3. (a) The differential equation is both separable and linear. Therefore, one way to obtain the solution to the initial-value problem is to integrate

$$\int \frac{1}{y}\,dy = \int t\,dt.$$

We obtain

$$\ln|y| = \frac{t^2}{2} + c$$

$$y = ke^{t^2/2}.$$

We determine the value of k using the initial condition $y(0) = 1$. Hence, $k = 1$, and the solution to the given initial-value problem is $y(t) = e^{t^2/2}$.

(b) To calculate y_{20}, we must apply Euler's method 20 times. Table 7.7 contains the results of a number of intermediate calculations.

(c) The total error e_{20} is the difference between the actual value $y(\sqrt{2}) = e$ and the approximate value $y_{20} = 2.51066$. Therefore, $e_{20} = 0.20762$.

(d) Table 7.8 contains the results of Euler's method and the corresponding total errors for $n = 1000$, $2000, \ldots, 6000$.

Table 7.7
Results of Euler's method

k	t_k	y_k
0	0	1
1	0.0707107	1
2	0.141421	1.005
3	0.212132	1.01505
4	0.282843	1.03028
5	0.353553	1.05088
⋮	⋮	⋮
10	0.707107	1.24797
⋮	⋮	⋮
19	1.3435	2.29284
20	1.41421	2.51066

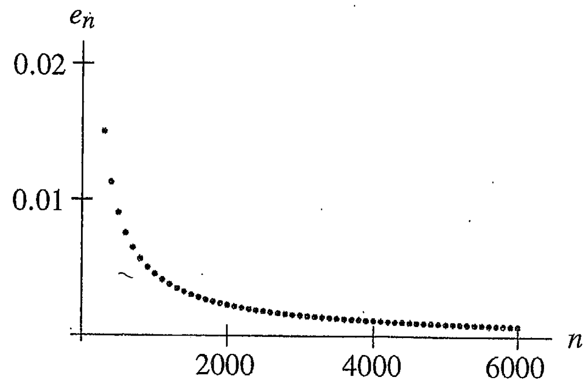
Table 7.8
Results of Euler's method and the corresponding total errors

n	y_n	e_n
1000	2.71376	0.00452218
2000	2.71602	0.00226316
3000	2.71677	0.00150923
4000	2.71715	0.0011321
5000	2.71738	0.000905762
6000	2.71753	0.000754848

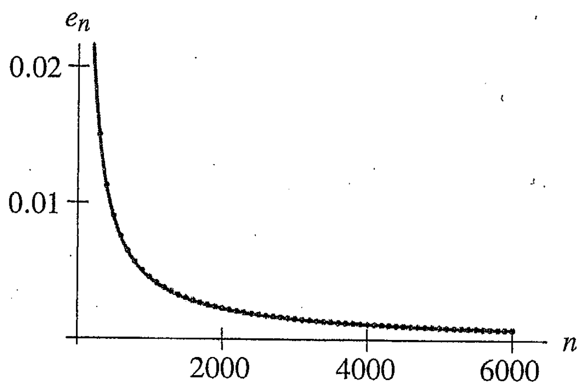
(e) Table 7.9 gives values of e_n for some n that are intermediate to the ones above in case that you want to double check the ones you have computed. Also, the graph of e_n as a function of n for $100 \le n \le 6000$ is given.

Table 7.9
Selected total errors

n	e_n
100	0.0444901
200	0.0224467
1400	0.00323182
3700	0.00122384
5600	0.000808748



(f) Our computer math system fits the data to the function $4.47023/n$. The following figure includes both the data and the graph of this function.



4. (a) The differential equation is separable. Therefore, we integrate

$$\int \frac{1}{y^2}\,dy = \int -1\,dt.$$

We obtain

$$-\frac{1}{y} = -t + c$$

$$y = \frac{1}{t+k}.$$

We determine the value of k using the initial condition $y(0) = 1/2$. Hence, $k = 2$, and the solution to the given initial-value problem is $y(t) = 1/(t+2)$.

(b) To calculate y_{20}, we must apply Euler's method 20 times. Table 7.10 contains the results of a number of intermediate calculations.

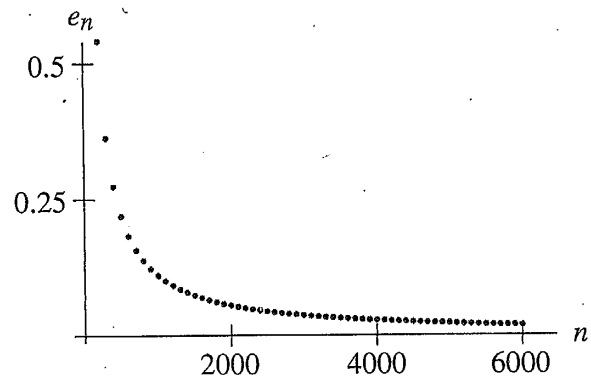
(c) The total error e_{20} is the difference between the actual value $y(2) = e$ and the approximate value $y_{20} = 2.51066$. Therefore, $e_{20} = 0.00444754$.

(d) Table 7.11 contains the results of Euler's method and the corresponding total errors for $n = 1000, 2000, \ldots, 6000$.

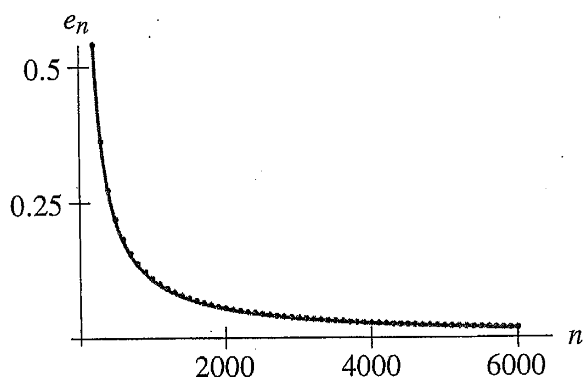
(e) Table 7.3 gives values of e_n for some n that are intermediate to the ones above in case that you want to double check the ones you have computed. Also, the graph of e_n as a function of n for $100 \le n \le 6000$ is given.

Table 7.15
Selected total errors

n	e_n
100	1.05955
200	0.540843
1400	0.0786686
3700	0.0298226
5600	0.0197119



(f) Our computer math system fits the data to the function $107.125/n$. The following figure includes both the data and the graph of this function.



6. For Euler's method, we assume that the error using n steps is of the form K/n for some constant K. Therefore, we assume that $e_{2000} = 0.000063 \approx K/2000$. We could use this observation to obtain an approximation for K, but unless we need to know K for some other reason, we can skip that step. Basically, we need only observe that, since Euler's method is a first-order numerical method, we need to increase the number of steps by a factor of 63 in order to lower the error by a factor of 63. Since $n = 2000$ yields an error of 0.000063, we must use $n = 63 \times 2000 = 126,000$ steps to obtain an error of 0.000001.

7. **(a)** The partial derivative $\partial f/\partial y$ of $f(t, y) = -2ty^2$ is $-4ty$. Thus, $M_2 = -4ty$.
The partial derivative $\partial f/\partial t$ of $f(t, y) = -2ty^2$ is $-2y^2$. Therefore,

$$M_1 = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f(t, y) = -2y^2 + (-4ty)(-2ty^2) = -2y^2 + 8t^2y^3.$$

(b) Since $f(t_0, y_0) = 0$, the result of the first step of Euler's method is the point $(t_1, y_1) = (0.02, 1)$. Consequently, to estimate the error, we compute the quantity M_1 at the point $(0.01, 1)$. We obtain $M_1 \approx -1.9992$.

Once we have M_1 at this point, we can estimate the error e_1 (which is the same as the truncation error) by computing $|M_1(\Delta t)^2/2|$. In this case, we obtain 0.00039984.

To calculate the actual error, we can compare the result y_1 of Euler's method with the value of the solution $y(0.2)$. Since we know that the solution is $1/(1+t^2)$, we have $y(0.2) = 0.9996$, and thus the actual error is 0.00039984. For this computation, note that the estimated error and the actual error essentially agree. (In order to see a difference in these two quantities, we had to do the calculations to 11 decimal places.)

(c) The second point (t_2, y_2) obtained from Euler's method is the point (0.04, 0.9992). For the second step, the estimated error is no longer simply the truncation error, so we must compute both M_1 and M_2. Evaluating M_1 at the point (0.03, 0.9996), we obtain $M_1 = -1.99121$. Evaluating M_2 at the point (0.02, 0.9996), we obtain $M_2 = -0.079968$.

Now to estimate the error in the second step, we use the approximation

$$e_k \approx (1 + M_2 \Delta t)\, e_{k-1} + M_1 \frac{(\Delta t)^2}{2}$$

and obtain $e_2 \approx 0.000797442$.

To compare this estimate to the actual error, we compute the value of the solution $y(0.04) = 0.998403$. Since $y_2 = 0.9992$, we see that the actual error e_2 is 0.000797444. Note that our estimate of e_2 and the true value of e_2 are very close.

(d) Table 7.16 gives the values of e_k and our estimates of e_k for $k = 10, 20, 30, \ldots, 100$ in case that you want to double check your computations.

Table 7.16
Selected total errors

k	t_k	e_k	estimated e_k
10	0.2	-0.00354477	-0.00354363
20	0.4	-0.00487796	-0.00486646
30	0.6	-0.00399684	-0.00396879
40	0.8	-0.00226733	-0.00223171
50	1.	-0.000714495	-0.000683793
60	1.2	0.000335713	0.000355439
70	1.4	0.000929043	0.000937796
80	1.6	0.00120617	0.00120657
90	1.8	0.00129161	0.0012866
100	2.	0.00127095	0.00126289

(e) Compare your plots with the Figures 7.5 and 7.6 in Section 7.1.

8. **(a)** The partial derivative $\partial f/\partial y$ of $f(t, y) = t - y^3$ is $-3y^2$. Thus, $M_2 = -3y^2$.
The partial derivative $\partial f/\partial t$ of $f(t, y) = t - y^3$ is 1. Therefore,

$$M_1 = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f(t, y) = 1 - 3y^2(t - y^3).$$

(b) The result of the first step of Euler's method is the point $(t_1, y_1) = (0.01, 0.99)$. Consequently, to estimate the error, we compute the quantity M_1 at the point (0.005, 0.995). We obtain $M_1 \approx 3.9109$.

both M_1 and M_2. Evaluating M_1 at the point $(0.045, 3.00135)$, we obtain $M_1 = 2.98002$. Evaluating M_2 at the point $(0.03, 3.00135)$, we obtain $M_2 = 0.0298785$.

Now to estimate the error in the second step, we use the approximation

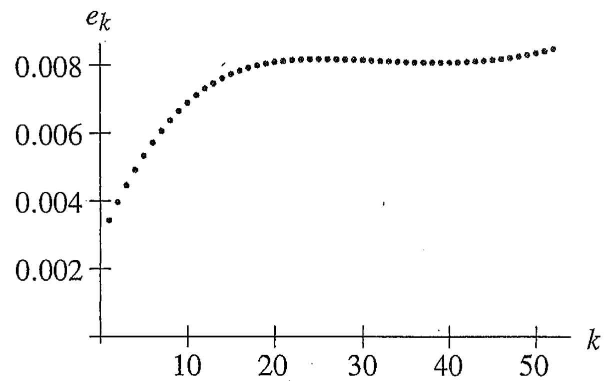
$$e_k \approx (1 + M_2 \Delta t)\, e_{k-1} + M_1 \frac{(\Delta t)^2}{2}$$

and obtain $e_2 \approx 0.00269115$.

(d) The following table gives the values of our estimates of e_k for $k = 10, 20, 30, \ldots, 100$ in case that you want to double check your computations. We also plot our estimates of the error as a function of k.

Table 7.18
Selected error estimates

k	t_k	estimated e_k
10	0.3	0.0123521
20	0.6	0.0138193
30	0.9	0.00135629
40	1.2	0.0109331
50	1.5	0.0121639
60	1.8	0.0119345
70	2.1	0.0129518
80	2.4	0.0164855
90	2.7	0.0240115
100	3.	0.0343504



10. The Taylor series for e^α is

$$1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \cdots.$$

Since all of the terms in this series are positive if $\alpha > 0$, we can truncate the series anywhere and obtain quantity that is less than e^α. In this case, we truncate the series after the first two terms and obtain

$$1 + \alpha < e^\alpha.$$

11. **(a)** The argument that justifies the inequality

$$e_1 \le M_1 \frac{(\Delta t)^2}{2}$$

is given on pages 631 and 632. In particular, the truncation error in the first step is given by Taylor's Theorem.

(b) The total error e_2 involved in the second step is discussed on pages 633 and 634. On the right-hand side of the inequality

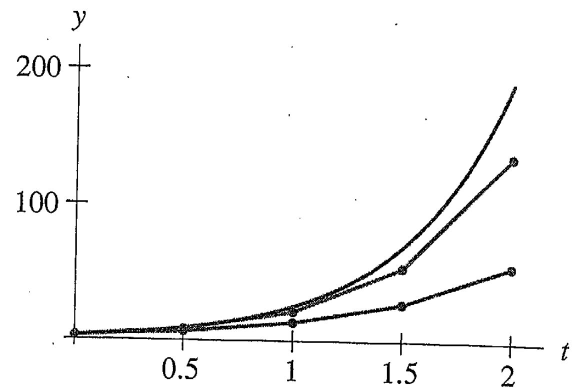
$$e_2 \le e_1 + M_2 e_1 \Delta t + M_1 \frac{(\Delta t)^2}{2},$$

EXERCISES FOR SECTION 7.2

1. Table 7.19 includes the approximate values y_k obtained using improved Euler's method. In addition to the results of improved Euler's method, we graph the results of Euler's method and the results obtained when we used a built-in numerical solver.

Table 7.19
Results of improved Euler's method

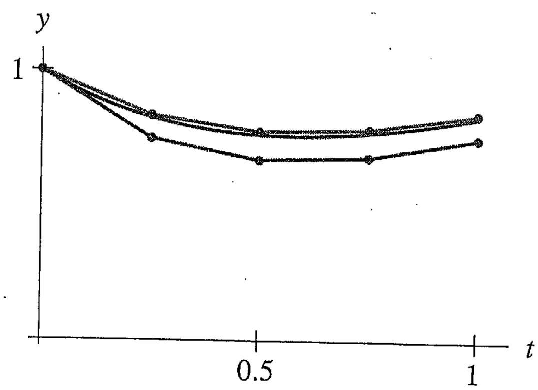
k	t_k	y_k
0	0.0	3.0000
1	0.5	8.2500
2	1.0	21.3750
3	1.5	54.1875
4	2.0	136.2187



2. Table 7.20 includes the approximate values y_k obtained using improved Euler's method. In addition to the results of improved Euler's method, we graph the results of Euler's method and the results obtained when we used a built-in numerical solver.

Table 7.20
Results of improved Euler's method

k	t_k	y_k
0	0.	1.
1	0.25	0.835937
2	0.5	0.776864
3	0.75	0.787177
4	1.	0.844469



3. Table 7.21 includes the approximate values y_k obtained using improved Euler's method. In addition to the results of improved Euler's method, we graph the results of Euler's method and the results obtained when we used a built-in numerical solver.

Table 7.21
Results of improved Euler's method

k	t_k	y_k	k	t_k	y_k
0	0.00	0.5	5	1.25	−1.70535
1	0.25	0.445801	6	1.50	−2.09616
2	0.50	0.103176	7	1.75	−2.39212
3	0.75	−0.501073	8	2.00	−2.63277
4	1.00	−1.16818			

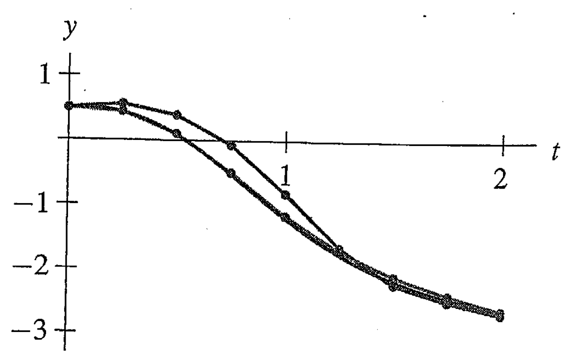
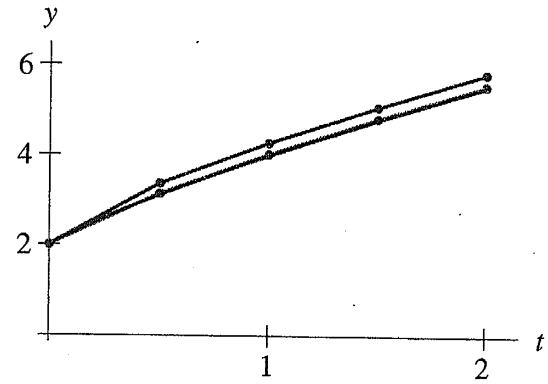


Table 7.24
Results of improved Euler's method

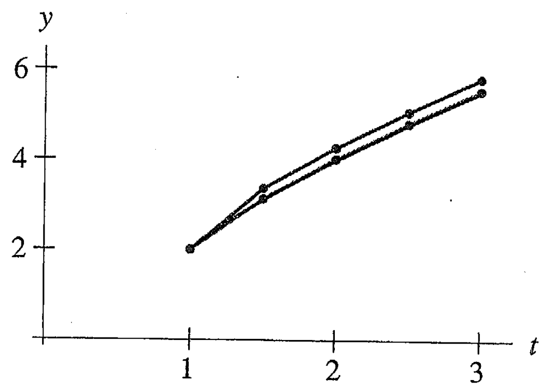
k	t_k	y_k
0	0.	2.
1	0.5	3.13301
2	1.	4.01452
3	1.5	4.80396
4	2.	5.54124



7. Table 7.25 includes the approximate values y_k obtained using improved Euler's method. Compare the results of this calculation with the results obtained in Exercise 6. In addition to the results of improved Euler's method, we graph the results of Euler's method and the results obtained when we used a built-in numerical solver.

Table 7.25
Results of improved Euler's method

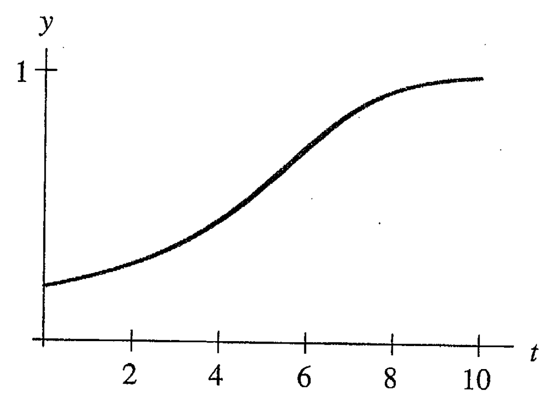
k	t_k	y_k
0	1.	2.
1	1.5	3.13301
2	2.	4.01452
3	2.5	4.80396
4	3.	5.54124



8. Table 7.26 includes the approximate values y_k obtained using improved Euler's method. In addition to the results of improved Euler's method, we graph the results of Euler's method and the results obtained when we used a built-in numerical solver. Note that it is basically impossible to distinguish the three graphs.

Table 7.26
Results of improved Euler's method

k	t_k	y_k
0	0.0	0.2
1	0.1	0.203
2	0.2	0.207
3	0.3	0.210
⋮	⋮	
99	9.9	0.989
100	10.0	0.990



(b) Table 7.30 contains the steps involved in applying improved Euler's method to this initial-value problem.

Table 7.30
Results of improved Euler's method

t_k	y_k
0.	1.
0.25	2.34375
0.5	4.9873
0.75	10.2711
1.	20.9178

Using the analytic solution, we know that the actual value of $y(1)$ is $(11e^3 + 1)/9$. Therefore, we can compute the error

$$e_4 = |y(1) - y_4| = 3.74227.$$

(c) If we want an approximation that is accurate to 0.0001, we need an improvement by a factor of

$$\frac{3.74227}{0.0001} = 37422.7.$$

Since improved Euler's method is a second-order numerical scheme, we expect to get that kind of improvement if we increase the number of steps by a factor of $\sqrt{37422.7}$. In other words, we compute the smallest integer larger than $4\sqrt{37422.7} = 773.798$. Using $n = 774$ steps, we get the approximate value $y_{774} = 24.6599$ and, consequently, an error $e_{774} \approx 0.000184$.

Unfortunately, this result is not within the tolerance specified in the statement of the exercise, and we must increase the number of steps once more. We want an additional improvement of a factor of

$$\frac{0.000184}{0.0001} = 1.84.$$

To determine our second choice for the number of steps, we compute $774\sqrt{1.84} = 1049.9$ Therefore, rather than 774 steps, we use 1050 steps. In this case, we get an error $e_{1050} \approx 0.0000999$.

13. **(a)** If we want an approximation that is accurate to 0.0001, we need an improvement by a factor of

$$\frac{0.000695}{0.0001} = 6.95.$$

Since improved Euler's method is a second-order numerical scheme, we expect to get that kind of improvement if we increase the number of steps by a factor of $\sqrt{6.95}$. In other words, we compute the smallest integer larger than $20\sqrt{6.95} = 52.7257$.

(b) Using $n = 53$ steps, we get the approximate value $y_{53} = 0.200095$.

(c) Consequently, the error e_{53} is the difference between the actual value $y(2) = 0.2$ and y_{53}. We get $e_{53} = 0.000095$.