Week 12

Monday November 10: Worksheet 28

**Laplace Transforms.** We shall be introduced to a new method for solving differential equations that involves an integral transform.

Reading:
Blanchard, Section 6.1

Homework #27:
Blanchard, Section 6.1: 2, 3, 4, 6, 7, 8, 12, 13, 16, 17.

Quiz:

Wednesday November 12: Worksheet 29

**Laplace Transforms and the Heaviside Function.** We shall continue our use of Laplace Transforms by considering discontinuous functions.

Reading:
Blanchard, 6.2

Homework #28:
Blanchard, Section 6.2: 3, 4, 5, 18, 21

Friday November 14: Worksheet 30

**Laplace Transform and Second-Order Equations.** We shall learn how to apply Laplace Transforms to solve second-order ordinary differential equations of the form $y'' + py' + qy = f(t)$.

Reading:
Blanchard, Section 6.3

Homework #29:
Blanchard, Section 6.3: 5, 6, 8, 9, 10, 15, 18, 27, 28.

Quiz:
Take-Home Quiz #7.
\[ \mathcal{L}[g_a(t)] = \int_0^\infty g_a(t) e^{-st} \, dt \]
\[ = \int_0^a \frac{t}{a} e^{-st} \, dt + \int_a^\infty e^{-st} \, dt \]

Using integration by parts with \( u = t \) and \( dv = e^{-st} \, dt \), we have \( du = dt \), \( v = -e^{-st}/s \) and
\[ \int_0^a \frac{t}{a} e^{-st} \, dt = \frac{1}{a} \int_0^a te^{-st} \, dt \]
\[ = \frac{1}{a} \left( -\frac{te^{-st}}{s} \bigg|_0^a - \int_0^a \frac{e^{-st}}{s} \, dt \right) \]
\[ = \frac{1}{a} \left( -\frac{ae^{-as}}{s} - \frac{1}{s^2} e^{-as} \bigg|_0^a \right) \]
\[ = \frac{1}{a} \left( -\frac{ae^{-as}}{s} - \frac{1}{s^2} (e^{-as} - 1) \right) \]
\[ = -\frac{e^{-as}}{s} - \frac{1}{as^2} (e^{-as} - 1). \]

Also,
\[ \int_a^\infty e^{-st} \, dt = \lim_{b \to \infty} \int_a^b e^{-st} \, dt \]
\[ = \lim_{b \to \infty} -\frac{1}{s} e^{-st} \bigg|_a^b \]
\[ = \lim_{b \to \infty} -\frac{1}{s} (e^{-sb} - e^{-as}) \]
\[ = \frac{1}{s} e^{-as}. \]

Therefore,
\[ \mathcal{L}[g_a(t)] = -\frac{e^{-as}}{s} - \frac{1}{as^2} (e^{-as} - 1) + \frac{1}{s} e^{-as} \]
\[ = \frac{1}{as^2} (1 - e^{-as}). \]

We have
\[ \mathcal{L}[e^{3t}] = \frac{1}{s - 3}, \]
so using the rule
\[ \mathcal{L}[u_a(t)y(t-a)] = e^{-as} \mathcal{L}[y(t)], \]
we determine that
\[ \mathcal{L}[u_2(t)e^{3(t-2)}] = \frac{e^{-2s}}{s - 3}. \]

The desired function is \( u_2(t)e^{3(t-2)}. \)
5. First use partial fractions to write

\[ \frac{1}{(s - 1)(s - 2)} = \frac{A}{s - 1} + \frac{B}{s - 2}. \]

Putting the right-hand side over a common denominator yields \( As - 2A + Bs - B = 1 \) which can be written as \((A + B)s + (-2A - B) = 1\). Thus, \( A + B = 0 \), and \(-2A - B = 1\). Solving for \( A \) and \( B \) yields \( A = -1 \) and \( B = 1 \), so

\[ \frac{1}{(s - 1)(s - 2)} = \frac{-1}{s - 1} + \frac{1}{s - 2}. \]

Now, as above

\[ \mathcal{L}[u_3(t)e^{2(t-3)}] = \frac{e^{-3s}}{s-2} \]

and

\[ \mathcal{L}[u_3(t)e^{t-3}] = \frac{e^{-3s}}{s-1} \]

and the desired function is

\[ u_3(t) \left( e^{2(t-3)} - e^{(t-3)} \right). \]

6. Using partial fractions, we write

\[ \frac{4}{s(s + 3)} = \frac{A}{s} + \frac{B}{s + 3}. \]

Hence, we must have \( As + 3A + Bs = 4 \) which can be written as \((A + B)s + 3A = 4\). So, \( A + B = 0 \), and \( 3A = 4 \). This gives us \( A = 4/3 \) and \( B = -4/3 \), so

\[ \frac{4}{s(s + 3)} = \frac{4/3}{s} - \frac{4/3}{s + 3}. \]

Applying the rules

\[ \mathcal{L}[u_2(t)] = \frac{e^{-2s}}{s} \]

and

\[ \mathcal{L}[u_2(t)e^{-3(t-2)}] = \frac{e^{-2s}}{s + 3}, \]

the desired function is

\[ y(t) = u_2(t) \left( \frac{4}{3} - \frac{4e^{-3(t-2)}}{3} \right) \]

or

\[ y(t) = \frac{4}{3} u_2(t) \left( 1 - e^{-3(t-2)} \right). \]
From the formula in Exercise 16, we see that we need only compute the integral \( \int_0^1 te^{-st} \, dt \). Using integration by parts (as in Exercise 2 of Section 6.1), we get
\[
\mathcal{L}[z] = \frac{1}{1 - e^{-s}} \left( \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} \right)
= \frac{1}{s^2} - \frac{e^{-s}}{s(1 - e^{-s})}.
\]

19. (a) Transforming both sides of the equation, we have \( \mathcal{L}[dy/dt] = -\mathcal{L}[y] + \mathcal{L}[w(t)] \), and using the result of Exercise 17, we get
\[
s\mathcal{L}[y] - y(0) = -\mathcal{L}[y] + \frac{1 - e^{-s}}{s(1 + e^{-s})}.
\]
Solving for \( \mathcal{L}[y] \) using the fact that \( y(0) = 0 \), we obtain
\[
\mathcal{L}[y] = \frac{1 - e^{-s}}{s(s + 1)(1 + e^{-s})}.
\]

(b) The function \( w(t) \) is alternatively 1 and \(-1\). While \( w(t) = 1 \), the solution decays exponentially toward \( y = 1 \). When \( w(t) \) changes to \(-1\), the solution then decays toward \( y = -1 \).

In fact, there is a periodic solution with initial condition \( y(0) = (1 - e)/(1 + e) \approx -0.462 \), and our solution tends toward this periodic solution as \( t \to \infty \).

20. (a) Transforming both sides of the equation, we have \( \mathcal{L}[dy/dt] = -\mathcal{L}[y] + \mathcal{L}[z(t)] \), and using the result of Exercise 18, we get
\[
s\mathcal{L}[y] - y(0) = -\mathcal{L}[y] + \frac{1}{s^2} - \frac{e^{-s}}{s(1 - e^{-s})}.
\]
Solving for \( \mathcal{L}[y] \) using the fact that \( y(0) = 0 \), we obtain
\[
\mathcal{L}[y] = \frac{1}{s^2(s + 1)} - \frac{e^{-s}}{s(s + 1)(1 - e^{-s})}.
\]

(b) We know that the solution to the unforced equation decays exponentially to 0, so the solution to the forced equation decays exponentially toward the forcing term. For \( 0 \leq t < 1 \), the forcing term is simply \( t \), and the solution is \( t - 1 + e^{-t} \). At \( t = 1 \), the forcing function \( z(t) \) jumps back to 0, yet \( y(1) = 1/e \). Thus the solution starts to decrease. However, for some value of \( t \) in the interval \( 1 \leq t \leq 2 \), \( z(t) = y(t) \), so the solution begins to increase again. At \( t = 2 \), the forcing function \( z(t) \) jumps back to zero again, and the solution begins to decrease again. Once again, for some value of \( t \) in the interval \( 2 \leq t \leq 3 \), \( z(t) = y(t) \), so the solution begins to increase again. This "oscillating" phenomenon repeats.

In fact, there is a periodic solution with initial condition \( y(0) = 1/(e - 1) \approx 0.582 \), and our solution tends toward this periodic solution as \( t \to \infty \).
XERCISES FOR SECTION 6.3

1. We use integration by parts twice to compute

\[ \mathcal{L}[\sin \omega t] = \int_0^\infty \sin \omega t \, e^{-st} \, dt. \]

First, letting \( u = \sin \omega t \) and \( dv = e^{-st} \, dt \), we get

\[ \mathcal{L}[\sin \omega t] = \sin \omega t \left. \frac{e^{-st}}{-s} \right|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} \omega \cos \omega t \, dt \]

\[ = \lim_{b \to \infty} \left[ \frac{e^{-st}}{-s} \sin \omega t \right]_0^b + \frac{\omega}{s} \int_0^\infty e^{-st} \cos \omega t \, dt \]

\[ = \frac{\omega}{s} \int_0^\infty e^{-st} \cos \omega t \, dt, \]

since the limit of \( e^{-sb} \sin \omega b \) is 0 as \( b \to \infty \) and \( s > 0 \).

Using integration by parts on

\[ \int_0^\infty e^{-st} \cos \omega t \, dt, \]

with \( u = \cos \omega t \) and \( dv = e^{-st} \, dt \), we get

\[ \int_0^\infty e^{-st} \cos \omega t \, dt = \frac{e^{-st}}{-s} \cos \omega t \bigg|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} (-\omega \sin \omega t) \, dt \]

\[ = \lim_{b \to \infty} \left[ \frac{e^{-st}}{-s} \cos \omega t \right]_0^b - \frac{\omega}{s} \int_0^\infty e^{-st} \sin \omega t \, dt \]

\[ = \lim_{b \to \infty} \left[ \frac{e^{-sb}}{-s} \cos \omega b \right] + \frac{1}{s} - \frac{\omega}{s} \int_0^\infty e^{-st} \sin \omega t \, dt \]

\[ = \frac{1}{s} - \frac{\omega}{s} \int_0^\infty e^{-st} \sin \omega t \, dt, \]
Then substituting back we have
\[ \mathcal{L}[e^{at} \cos \omega t] = \frac{s - a}{(s - a)^2 + \omega^2}. \]

Using the formula
\[ \mathcal{L} \left[ \frac{d^2 y}{dt^2} \right] = s^2 \mathcal{L}[y] - y'(0) - sy(0), \]
and the linearity of the Laplace transform, we get that
\[ s^2 \mathcal{L}[y] - y'(0) - sy(0) + \omega^2 \mathcal{L}[y] = 0. \]
Substituting the initial conditions and solving for \( \mathcal{L}[y] \) gives
\[ \mathcal{L}[y] = \frac{s}{s^2 + \omega^2}. \]

Since
\[ \mathcal{L}[\cos \omega t] = \frac{s}{s^2 + \omega^2}, \]
we can compute that
\[ \frac{d}{d\omega} \mathcal{L}[\cos \omega t] = \frac{-s(2\omega)}{(s^2 + \omega^2)^2} = \frac{-2\omega s}{(s^2 + \omega^2)^2}, \]
but
\[ \frac{d}{d\omega} \mathcal{L}[\cos \omega t] = \mathcal{L} \left[ \frac{d}{d\omega} \cos \omega t \right] = \mathcal{L}[-t \sin \omega t]. \]

We can bring the derivative with respect to \( \omega \) inside the Laplace transform because the Laplace transform is an integral with respect to \( t \), that is,
\[ \frac{d}{d\omega} \mathcal{L}[\cos \omega t] = \frac{d}{d\omega} \int_0^\infty \cos \omega t e^{-st} \, dt = \int_0^\infty \frac{d}{d\omega} (\cos \omega t e^{-st}) \, dt. \]
Canceling the minus signs on left and right gives
\[ \mathcal{L}[t \sin \omega t] = \frac{2\omega s}{(s^2 + \omega^2)^2}. \]

7. Since
\[ \mathcal{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}, \]
we can compute that
\[ \frac{d}{d\omega} \mathcal{L}[\sin \omega t] = \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}, \]
but
\[ \frac{d}{d\omega} \mathcal{L}[\sin \omega t] = \mathcal{L} \left[ \frac{d}{d\omega} \sin \omega t \right] = \mathcal{L}[t \cos \omega t]. \]
So
\[ \mathcal{L}[t \cos \omega t] = \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}. \]
8. We need to compute
\[ \mathcal{L}[te^{at}] = \int_0^\infty te^{at}e^{-st} dt. \]

We can do this using the hint, by differentiating \( \mathcal{L}[e^{at}] \) with respect to \( a \). Another method is to write
\[ \mathcal{L}[te^{at}] = \int_0^\infty te^{at}e^{-st} dt = \int_0^\infty te^{-(s-a)t} dt = \int_0^\infty te^{-rt} dt \]
where \( r = s - a \). The last integral is the Laplace transform of \( t \) using \( r \) as the new independent variable. Hence, from the table we have
\[ \int_0^\infty te^{-rt} dt = \frac{1}{r^2}. \]
Substituting back \( r = s - a \) we have
\[ \mathcal{L}[te^{at}] = \frac{1}{(s-a)^2}. \]

9. From Exercise 10, we know that
\[ \mathcal{L}[te^{at}] = \frac{1}{(s-a)^2}. \]
Differentiating both sides of this formula with respect to \( a \) gives
\[ \frac{d}{da} \mathcal{L}[te^{at}] = \mathcal{L}\left[ \frac{d}{da} te^{ta} \right] = \mathcal{L}[t^2e^{at}] \]
while
\[ \frac{d}{da} \frac{1}{(s-a)^2} = \frac{2}{(s-a)^3}. \]
Hence,
\[ \mathcal{L}[t^2e^{at}] = \frac{2}{(s-a)^3}. \]

10. Using the results of Exercise 9, we can work by induction on \( n \), with induction hypothesis
\[ \mathcal{L}[t^n e^{at}] = \frac{n!}{(s-a)^{n+1}}. \]
Alternately, we can compute
\[ \mathcal{L}[t^n e^{at}] = \int_0^\infty t^n e^{at}e^{-st} dt = \int_0^\infty t^n e^{-(s-a)t} dt = \int_0^\infty t^n e^{-rt} dt \]
where \( r = s - a \). Now the last integral is the Laplace transform of \( t^n \) using \( r \) as the independent variable, so
\[ \int_0^\infty t^n e^{-rt} dt = \frac{n!}{r^{n+1}} \]
from the table. Hence, substituting \( r = s - a \) we have
\[ \mathcal{L}[t^n e^{at}] = \frac{n!}{(s-a)^{n+1}}. \]
11. In this case, \( b = 2 \), and \((s + b/2)^2 = (s + 1)^2 = s^2 + 2s + 1\), so \( s^2 + 2s + 10 = (s + 1)^2 + 3^2 \).

12. In this case, \( b = -4 \), and \((s + b/2)^2 = (s - 2)^2 = s^2 - 4s + 4\), so \( s^2 - 4s + 5 = (s - 2)^2 + 1^2 \).

13. In this case, \( b = 1 \), and \((s + b/2)^2 = (s + 1/2)^2 = s^2 + s + 1/4\), so \( s^2 + s + 1 = (s + 1/2)^2 + 3/4 = (s + 1/2)^2 + (\sqrt{3}/2)^2 \).

14. In this case, \( b = 6 \), and \((s + b/2)^2 = (s + 3)^2 = s^2 + 6s + 9\), so \( s^2 + 6s + 10 = (s + 3)^2 + 1^2 \).

15. In Exercise 11, we completed the square and obtained \( s^2 + 2s + 10 = (s + 1)^2 + 3^2 \), so

\[
\mathcal{L}^{-1}\left[\frac{1}{s^2 + 2s + 10}\right] = \mathcal{L}^{-1}\left[\frac{1}{(s + 1)^2 + 3^2}\right] = \frac{1}{3}\mathcal{L}^{-1}\left[\frac{3}{(s + 1)^2 + 3^2}\right] = \frac{1}{3}e^{-t}\sin 3t.
\]

16. In Exercise 12, we completed the square and obtained \( s^2 - 4s + 5 = (s - 2)^2 + 1^2 \), so

\[
\mathcal{L}^{-1}\left[\frac{s}{s^2 - 4s + 5}\right] = \mathcal{L}^{-1}\left[\frac{s}{(s - 2)^2 + 1^2}\right] = \mathcal{L}^{-1}\left[\frac{s - 2}{(s - 2)^2 + 1^2}\right] + \mathcal{L}^{-1}\left[\frac{2}{(s - 2)^2 + 1^2}\right] = e^{2t}\cos t + e^{2t}(2\sin t) = e^{2t}(\cos t + 2\sin t).
\]

17. In Exercise 13, we completed the square and obtained \( s^2 + s + 1 = (s + 1/2)^2 + (\sqrt{3}/2)^2 \), so

\[
\frac{2s + 3}{s^2 + s + 1} = \frac{2s + 3}{(s + 1/2)^2 + (\sqrt{3}/2)^2}.
\]

We want to put this fraction in the right form so that we can use the formulas for \(\mathcal{L}[e^{at}\cos \omega t]\) and \(\mathcal{L}[e^{at}\sin \omega t]\). We see that

\[
\frac{2s + 3}{(s + 1/2)^2 + (\sqrt{3}/2)^2} = \frac{2s + 1}{(s + 1/2)^2 + (\sqrt{3}/2)^2} + \frac{2}{(s + 1/2)^2 + (\sqrt{3}/2)^2} = \frac{2(s + 1/2)}{(s + 1/2)^2 + (\sqrt{3}/2)^2} + \frac{(4/\sqrt{3})(\sqrt{3}/2)}{(s + 1/2)^2 + (\sqrt{3}/2)^2}.
\]

So

\[
\mathcal{L}^{-1}\left[\frac{2s + 3}{s^2 + s + 1}\right] = 2\mathcal{L}^{-1}\left[\frac{(s + 1/2)}{(s + 1/2)^2 + (\sqrt{3}/2)^2}\right] + \frac{4}{\sqrt{3}}\mathcal{L}^{-1}\left[\frac{\sqrt{3}/2}{(s + 1/2)^2 + (\sqrt{3}/2)^2}\right] = 2e^{-t/2}\cos \left(\frac{\sqrt{3}}{2}t\right) + \frac{4}{\sqrt{3}}e^{-t/2}\sin \left(\frac{\sqrt{3}}{2}t\right).
\]
In Exercise 14, we completed the square and obtained \( s^2 + 6s + 10 = (s + 3)^2 + 1^2 \), so

\[
\frac{s + 1}{s^2 + 6s + 10} = \frac{s + 1}{(s + 3)^2 + 1^2}.
\]

We want to put this fraction in the right form so that we can use the formulas for \( \mathcal{L}[e^{at} \cos \omega t] \) and \( \mathcal{L}[e^{at} \sin \omega t] \). We see that

\[
\frac{s + 1}{(s + 3)^2 + 1^2} = \frac{s + 3}{(s + 3)^2 + 1^2} - \frac{2}{(s + 3)^2 + 1^2}.
\]

So

\[
\mathcal{L}^{-1}\left[\frac{s + 1}{s^2 + 6s + 10}\right] = e^{-3t} \cos t - 2e^{-3t} \sin t.
\]

9. We compute

\[
\mathcal{L}\left[e^{(a+ib)t}\right] = \int_0^\infty e^{(a+ib)t} e^{-st} \, dt
\]

\[
= \int_0^\infty e^{-(s-(a+ib))t} \, dt
\]

\[
= \frac{1}{s - (a + ib)} \left( \lim_{u \to \infty} \left[ e^{-(s-a)u} e^{-ibu} \right] - 1 \right).
\]

The limit is zero as long as \( s > a \). Hence,

\[
\mathcal{L}\left[e^{(a+ib)t}\right] = \frac{1}{s - (a + ib)}
\]

if \( s > a \) and undefined otherwise. This is the same formula as for real exponentials. It can also be written

\[
\mathcal{L}\left[e^{(a+ib)t}\right] = \frac{s - a + ib}{(s-a)^2 + b^2}.
\]

20. This follows from linearity:

\[
\mathcal{L}[y] = \mathcal{L}[y_{re} + iy_{im}]
\]

\[
= \int_0^\infty (y_{re} + iy_{im}) e^{-st} \, dt
\]

\[
= \int_0^\infty y_{re}(t) e^{-st} \, dt + i \int_0^\infty y_{im}(t) e^{-st} \, dt
\]

\[
= \mathcal{L}[y_{re}] + i \mathcal{L}[y_{im}].
\]
Taking inverse Laplace transforms of the right-hand side gives
\[
\left(1 - \frac{2}{\sqrt{3}} i\right) e^{(-1+i\sqrt{3})t/2} + \left(1 + \frac{2}{\sqrt{3}} i\right) e^{(-1-i\sqrt{3})t/2}.
\]

Using Euler's formula to replace the complex exponentials and simplifying yields
\[
2e^{-t/2} \cos \left(\frac{\sqrt{3}}{2} t\right) + \frac{4}{\sqrt{3}} e^{-t/2} \sin \left(\frac{\sqrt{3}}{2} t\right).
\]

26. Using the quadratic formula, we find the roots of the denominator are \(-3 \pm i\) so the denominator can be factored
\[
s^2 + 6s + 10 = (s - (-3 + i))(s - (-3 - i)).
\]
The partial fractions decomposition is
\[
\frac{s + 1}{s^2 + 6s + 10} = \frac{A}{s - (-3 - i)} + \frac{B}{s - (-3 + i)},
\]
which leads to the equations
\[
\begin{aligned}
A + B &= 1 \\
(3 - i)A + (3 + i)B &= 1.
\end{aligned}
\]

Solving, we find \(A = \frac{1}{2} - i\) and \(B = \frac{1}{2} + i\), so
\[
\frac{s + 1}{s^2 + 6s + 10} = \frac{\frac{1}{2} - i}{s - (-3 - i)} + \frac{\frac{1}{2} + i}{s - (-3 + i)}.
\]

Taking inverse Laplace transform of the right-hand side gives
\[
\left(\frac{1}{2} - i\right) e^{(-3-i)t} + \left(\frac{1}{2} + i\right) e^{(-3+i)t}
\]
and using Euler's formula and simplifying gives
\[
e^{-3t} \cos t - 2e^{-3t} \sin t.
\]

27. (a) Taking the Laplace transform of both sides of the equation, we obtain
\[
\mathcal{L} \left[\frac{d^2 y}{dt^2}\right] + 4 \mathcal{L}[y] = \frac{8}{s},
\]
and using the fact that \(\mathcal{L}[d^2 y/dt^2] = s^2 \mathcal{L}[y] - sy(0) - y'(0)\), we have
\[
(s^2 + 4) \mathcal{L}[y] - sy(0) - y'(0) = \frac{8}{s}.
\]
(b) Substituting the initial conditions yields

\[(s^2 + 4)\mathcal{L}[y] - 11s - 5 = \frac{8}{s},\]

and solving for \(\mathcal{L}[y]\) we get

\[\mathcal{L}[y] = \frac{11s + 5}{s^2 + 4} + \frac{8}{s(s^2 + 4)}.\]

The partial fractions decomposition of \(8/(s(s^2 + 4))\) is

\[\frac{8}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4}.\]

Putting the right-hand side over a common denominator gives us

\[(A + B)s^2 + Cs + 4A = 8,\]

and consequently, \(A = 2\), \(B = -2\), and \(C = 0\). In other words,

\[\frac{8}{s(s^2 + 4)} = \frac{2}{s} + \frac{-2s}{s^2 + 4}.\]

We obtain

\[\mathcal{L}[y] = \frac{2}{s} + \frac{9s + 5}{s^2 + 4}.\]

(c) To take the inverse Laplace transform, we rewrite \(\mathcal{L}[y]\) in the form

\[\mathcal{L}[y] = \frac{2}{s} + 9 \left( \frac{s}{s^2 + 4} \right) + 5 \left( \frac{2}{s^2 + 4} \right).\]

Therefore, \(y(t) = 2 + 9 \cos 2t + \frac{5}{2} \sin 2t.\)

28. (a) Taking the Laplace transform of both sides of the equation, we obtain

\[\mathcal{L} \left[ \frac{d^2 y}{dt^2} \right] - \mathcal{L}[y] = \frac{1}{s - 2},\]

and using the fact that \(\mathcal{L}[d^2 y/dt^2] = s^2 \mathcal{L}[y] - sy(0) - y'(0),\) we have

\[(s^2 - 1)\mathcal{L}[y] - sy(0) - y'(0) = \frac{1}{s - 2}.\]

(b) Substituting the initial conditions yields

\[(s^2 - 1)\mathcal{L}[y] - s + 1 = \frac{1}{s - 2},\]

and solving for \(\mathcal{L}[y]\) we get

\[\mathcal{L}[y] = \frac{1}{s + 1} + \frac{1}{(s - 2)(s^2 - 1)}.\]
Using the partial fractions decomposition
\[ \frac{1}{(s - 2)(s^2 - 1)} = \frac{1}{3} \frac{1}{s - 2} + \frac{-1/2}{s - 1} + \frac{1/6}{s + 1}, \]
we obtain
\[ \mathcal{L}[y] = \frac{1}{3} \frac{1}{s - 2} + \frac{-1/2}{s - 1} + \frac{7/6}{s + 1}. \]

(c) Taking the inverse Laplace transform, we have
\[ y(t) = \frac{1}{3} e^{2t} - \frac{1}{2} e^t + \frac{7}{6} e^{-t}. \]

29. (a) Taking the Laplace transform of both sides of the equation, we obtain
\[ \mathcal{L} \left[ \frac{d^2 y}{dt^2} \right] - 4 \mathcal{L} \left[ \frac{dy}{dt} \right] + 5 \mathcal{L}[y] = \frac{2}{s - 1}, \]
and using the formulas for \( \mathcal{L}[dy/dt] \) and \( \mathcal{L}[d^2 y/dt^2] \) in terms of \( \mathcal{L}[y] \), we have
\[ (s^2 - 4s + 5) \mathcal{L}[y] - sy(0) - y'(0) + 4y(0) = \frac{2}{s - 1}. \]

(b) Substituting the initial conditions yields
\[ (s^2 - 4s + 5) \mathcal{L}[y] - 3s + 11 = \frac{2}{s - 2}, \]
and solving for \( \mathcal{L}[y] \) we get
\[ \mathcal{L}[y] = \frac{3s - 11}{s^2 - 4s + 5} + \frac{2}{(s - 1)(s^2 - 4s + 5)}. \]
Using the partial fractions decomposition
\[ \frac{2}{(s - 1)(s^2 - 4s + 5)} = \frac{1}{s - 1} + \frac{-s + 3}{s^2 - 4s + 5}, \]
we obtain
\[ \mathcal{L}[y] = \frac{1}{s - 1} + \frac{2s - 8}{s^2 - 4s + 5}. \]

(c) In order to compute the inverse Laplace transform, we first write
\[ s^2 - 4s + 5 = (s - 2)^2 + 1 \]
by completing the square, and then we write
\[ \frac{2s - 8}{s^2 - 4s + 5} = \frac{2(s - 2)}{(s - 2)^2 + 1} - \frac{4}{(s - 2)^2 + 1}. \]
Taking the inverse Laplace transform, we have
\[ y(t) = e^t + 2e^{2t} \cos t - 4e^{2t} \sin t. \]