Differential Equations

Math 341 Fall 2008

Fowler 307 MWF 2:30pm - 3:25pm
http://faculty.oxy.edu/ron/math/341/08/

Week 12

Monday November 10 : *Worksheet 28*

Laplace Transforms. We shall be introduced to a new method for solving differential equations that involves an integral transform.

Reading:

Blanchard, Section 6.1

Homework #27:

Blanchard, Section 6.1: 2, 3, 4, 6, 7, 8, 12, 13, 16, 17.

Quiz:

Wednesday November 12 : *Worksheet 29*

Laplace Transforms and the Heaviside Function. We shall continue our use of Laplace Transforms by considering discontinuous functions.

Reading:

Blanchard, 6.2

Homework #28:

Blanchard, Section 6.2: 3, 4, 5, 18, 21 → NO 21 so I copied 20.

Friday November 14 : *Worksheet 30*

Laplace Transform and Second-Order Equations. We shall learn how to apply Laplace Transforms to solve second-order ordinary differential equations of the form $y'' + py' + qy = f(t)$.

Reading:

Blanchard, Section 6.3

Homework #29:

Blanchard, Section 6.3: 5, 6, 8, 9, 10, 15, 18, 27, 28.

Quiz:

Take-Home Quiz #7.

3.

$$\begin{aligned}\mathcal{L}[g_a(t)] &= \int_0^\infty g_a(t)\, e^{-st}\, dt \\ &= \int_0^a \frac{t}{a} e^{-st}\, dt + \int_a^\infty e^{-st}\, dt\end{aligned}$$

Using integration by parts with $u = t$ and $dv = e^{-st}dt$, we have $du = dt$, $v = -e^{-st}/s$ and

$$\begin{aligned}\int_0^a \frac{t}{a} e^{-st}\, dt &= \frac{1}{a}\int_0^a te^{-st}\, dt \\ &= \frac{1}{a}\left(-\frac{te^{-st}}{s}\Big|_0^a - \int_0^a -\frac{e^{-st}}{s}\, dt\right) \\ &= \frac{1}{a}\left(-\frac{ae^{-as}}{s} - \frac{1}{s^2}e^{-st}\Big|_0^a\right) \\ &= \frac{1}{a}\left(-\frac{ae^{-as}}{s} - \frac{1}{s^2}(e^{-as} - 1)\right) \\ &= -\frac{e^{-as}}{s} - \frac{1}{as^2}\left(e^{-as} - 1\right).\end{aligned}$$

Also,

$$\begin{aligned}\int_a^\infty e^{-st}\, dt &= \lim_{b\to\infty}\int_a^b e^{-st}\, dt \\ &= \lim_{b\to\infty} -\frac{1}{s}e^{-st}\Big|_a^b \\ &= \lim_{b\to\infty} -\frac{1}{s}\left(e^{-sb} - e^{-as}\right) \\ &= \frac{1}{s}e^{-as}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{L}[g_a(t)] &= -\frac{e^{-as}}{s} - \frac{1}{as^2}\left(e^{-as} - 1\right) + \frac{1}{s}e^{-as} \\ &= \frac{1}{as^2}\left(1 - e^{-as}\right).\end{aligned}$$

4. We have

$$\mathcal{L}[e^{3t}] = \frac{1}{s-3},$$

so using the rule

$$\mathcal{L}[u_a(t)y(t-a)] = e^{-as}\mathcal{L}[y(t)],$$

we determine that

$$\mathcal{L}[u_2(t)e^{3(t-2)}] = \frac{e^{-2s}}{s-3}.$$

The desired function is $u_2(t)e^{3(t-2)}$.

5. First use partial fractions to write

$$\frac{1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}.$$

Putting the right-hand side over a common denominator yields $As - 2A + Bs - B = 1$ which can be written as $(A+B)s + (-2A - B) = 1$. Thus, $A + B = 0$, and $-2A - B = 1$. Solving for A and B yields $A = -1$ and $B = 1$, so

$$\frac{1}{(s-1)(s-2)} = \frac{-1}{s-1} + \frac{1}{s-2}.$$

Now, as above

$$\mathcal{L}[u_3(t)e^{2(t-3)}] = \frac{e^{-3s}}{s-2}$$

and

$$\mathcal{L}[u_3(t)e^{t-3}] = \frac{e^{-3s}}{s-1}$$

and the desired function is

$$u_3(t)\left(e^{2(t-3)} - e^{(t-3)}\right).$$

6. Using partial fractions, we write

$$\frac{4}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}.$$

Hence, we must have $As + 3A + Bs = 4$ which can be written as $(A+B)s + 3A = 4$. So, $A + B = 0$, and $3A = 4$. This gives us $A = 4/3$ and $B = -4/3$, so

$$\frac{4}{s(s+3)} = \frac{4/3}{s} - \frac{4/3}{s+3}.$$

Applying the rules

$$\mathcal{L}[u_2(t)] = \frac{e^{-2s}}{s}$$

and

$$\mathcal{L}[u_2(t)e^{-3(t-2)}] = \frac{e^{-2s}}{s+3},$$

the desired function is

$$y(t) = u_2(t)\left(\frac{4}{3} - \frac{4e^{-3(t-2)}}{3}\right)$$

or

$$y(t) = \frac{4}{3}u_2(t)\left(1 - e^{-3(t-2)}\right).$$

18. From the formula in Exercise 16, we see that we need only compute the integral $\int_0^1 te^{-st}\,dt$. Using integration by parts (as in Exercise 2 of Section 6.1), we get

$$\begin{aligned}\mathcal{L}[z] &= \frac{1}{1-e^{-s}}\left(\frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}\right)\\ &= \frac{1}{s^2} - \frac{e^{-s}}{s(1-e^{-s})}.\end{aligned}$$

19. **(a)** Transforming both sides of the equation, we have $\mathcal{L}\,[dy/dt] = -\mathcal{L}[y] + \mathcal{L}[w(t)]$, and using the result of Exercise 17, we get

$$s\mathcal{L}[y] - y(0) = -\mathcal{L}[y] + \frac{1-e^{-s}}{s(1+e^{-s})}.$$

Solving for $\mathcal{L}[y]$ using the fact that $y(0) = 0$, we obtain

$$\mathcal{L}[y] = \frac{1-e^{-s}}{s(s+1)(1+e^{-s})}.$$

(b) The function $w(t)$ is alternatively 1 and -1. While $w(t) = 1$, the solution decays exponentially toward $y = 1$. When $w(t)$ changes to -1, the solution then decays toward $y = -1$.

In fact, there is a periodic solution with initial condition $y(0) = (1-e)/(1+e) \approx -0.462$, and our solution tends toward this periodic solution as $t \to \infty$.

20. **(a)** Transforming both sides of the equation, we have $\mathcal{L}\,[dy/dt] = -\mathcal{L}[y] + \mathcal{L}[z(t)]$, and using the result of Exercise 18, we get

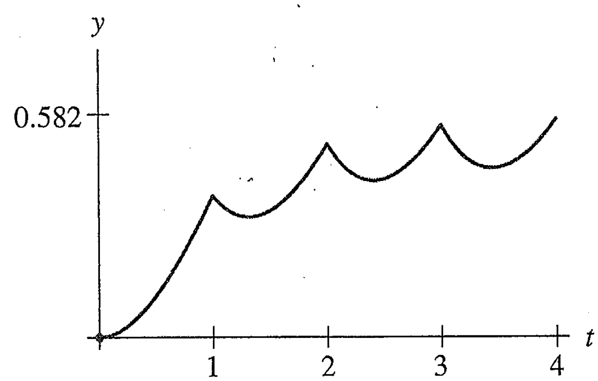
$$s\mathcal{L}[y] - y(0) = -\mathcal{L}[y] + \frac{1}{s^2} - \frac{e^{-s}}{s(1-e^{-s})}.$$

Solving for $\mathcal{L}[y]$ using the fact that $y(0) = 0$, we obtain

$$\mathcal{L}[y] = \frac{1}{s^2(s+1)} - \frac{e^{-s}}{s(s+1)(1-e^{-s})}.$$

(b) We know that the solution to the unforced equation decays exponentially to 0, so the solution to the forced equation decays exponentially toward the forcing term. For $0 \le t < 1$, the forcing term is simply t, and the solution is $t - 1 + e^{-t}$. At $t = 1$, the forcing function $z(t)$ jumps back to 0, yet $y(1) = 1/e$. Thus the solution starts to decrease. However, for some value of t in the interval $1 \le t \le 2$, $z(t) = y(t)$, so the solution begins to increase again. At $t = 2$, the forcing function $z(t)$ jumps back to zero again, and the solution begins to decrease again. Once again, for some value of t in the interval $2 \le t \le 3$, $z(t) = y(t)$, so the solution begins to increase again. This "oscillating" phenomenon repeats.

In fact, there is a periodic solution with initial condition $y(0) = 1/(e-1) \approx 0.582$, and our solution tends toward this periodic solution as $t \to \infty$.



XERCISES FOR SECTION 6.3

1. We use integration by parts twice to compute

$$\mathcal{L}[\sin\omega t] = \int_0^\infty \sin\omega t\, e^{-st}\, dt.$$

First, letting $u = \sin\omega t$ and $dv = e^{-st}\, dt$, we get

$$\begin{aligned}\mathcal{L}[\sin\omega t] &= \sin\omega t \left.\frac{e^{-st}}{-s}\right|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\omega\cos\omega t\, dt \\ &= \lim_{b\to\infty}\left[\frac{e^{-st}}{-s}\sin\omega t\Big|_0^b\right] + \frac{\omega}{s}\int_0^\infty e^{-st}\cos\omega t\, dt \\ &= \frac{\omega}{s}\int_0^\infty e^{-st}\cos\omega t\, dt,\end{aligned}$$

since the limit of $e^{-sb}\sin\omega b$ is 0 as $b\to\infty$ and $s > 0$.

Using integration by parts on

$$\int_0^\infty e^{-st}\cos\omega t\, dt,$$

with $u = \cos\omega t$ and $dv = e^{-st}\, dt$, we get

$$\begin{aligned}\int_0^\infty e^{-st}\cos\omega t\, dt &= \frac{e^{-st}}{-s}\cos\omega t\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}(-\omega\sin\omega t)\, dt \\ &= \lim_{b\to\infty}\left[\frac{e^{-st}}{-s}\cos\omega t\Big|_0^b\right] - \frac{\omega}{s}\int_0^\infty e^{-st}\sin\omega t\, dt \\ &= \lim_{b\to\infty}\left[\frac{e^{-sb}}{-s}\cos\omega b\right] + \frac{1}{s} - \frac{\omega}{s}\int_0^\infty e^{-st}\sin\omega t\, dt \\ &= \frac{1}{s} - \frac{\omega}{s}\int_0^\infty e^{-st}\sin\omega t\, dt,\end{aligned}$$

Then substituting back we have

$$\mathcal{L}[e^{at}\cos\omega t] = \frac{s-a}{(s-a)^2+\omega^2}.$$

5. Using the formula

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] = s^2\mathcal{L}[y] - y'(0) - sy(0),$$

and the linearity of the Laplace transform, we get that

$$s^2\mathcal{L}[y] - y'(0) - sy(0) + \omega^2\mathcal{L}[y] = 0.$$

Substituting the initial conditions and solving for $\mathcal{L}[y]$ gives

$$\mathcal{L}[y] = \frac{s}{s^2+\omega^2}.$$

6. Since

$$\mathcal{L}[\cos\omega t] = \frac{s}{s^2+\omega^2},$$

we can compute that

$$\frac{d}{d\omega}\mathcal{L}[\cos\omega t] = \frac{-s(2\omega)}{(s^2+\omega^2)^2} = \frac{-2\omega s}{(s^2+\omega^2)^2},$$

but

$$\frac{d}{d\omega}\mathcal{L}[\cos\omega t] = \mathcal{L}\left[\frac{d}{d\omega}\cos\omega t\right] = \mathcal{L}[-t\sin\omega t].$$

We can bring the derivative with respect to ω inside the Laplace transform because the Laplace transform is an integral with respect to t, that is,

$$\frac{d}{d\omega}\mathcal{L}[\cos\omega t] = \frac{d}{d\omega}\int_0^\infty \cos\omega t\, e^{-st}\,dt = \int_0^\infty \frac{d}{d\omega}\left(\cos\omega t\, e^{-st}\right)\,dt.$$

Canceling the minus signs on left and right gives

$$\mathcal{L}[t\sin\omega t] = \frac{2\omega s}{(s^2+\omega^2)^2}.$$

7. Since

$$\mathcal{L}[\sin\omega t] = \frac{\omega}{s^2+\omega^2},$$

we can compute that

$$\frac{d}{d\omega}\mathcal{L}[\sin\omega t] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2},$$

but

$$\frac{d}{d\omega}\mathcal{L}[\sin\omega t] = \mathcal{L}\left[\frac{d}{d\omega}\sin\omega t\right] = \mathcal{L}[t\cos\omega t].$$

So

$$\mathcal{L}[t\cos\omega t] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}.$$

8. We need to compute

$$\mathcal{L}[te^{at}] = \int_0^\infty te^{at}e^{-st}\,dt.$$

We can do this using the hint, by differentiating $\mathcal{L}[e^{at}]$ with respect to a. Another method is to write

$$\mathcal{L}[te^{at}] = \int_0^\infty te^{at}e^{-st}\,dt = \int_0^\infty te^{-(s-a)t}\,dt = \int_0^\infty te^{-rt}\,dt$$

where $r = s - a$. The last integral is the Laplace transform of t using r as the new independent variable. Hence, from the table we have

$$\int_0^\infty te^{-rt}\,dt = \frac{1}{r^2}.$$

Substituting back $r = s - a$ we have

$$\mathcal{L}[te^{at}] = \frac{1}{(s-a)^2}.$$

9. From Exercise 10, we know that

$$\mathcal{L}[te^{at}] = \frac{1}{(s-a)^2}.$$

Differentiating both sides of this formula with respect to a gives

$$\frac{d}{da}\mathcal{L}[te^{at}] = \mathcal{L}\left[\frac{d}{da}te^{ta}\right] = \mathcal{L}[t^2e^{at}]$$

while

$$\frac{d}{da}\frac{1}{(s-a)^2} = \frac{2}{(s-a)^3}.$$

Hence,

$$\mathcal{L}[t^2e^{at}] = \frac{2}{(s-a)^3}.$$

10. Using the results of Exercise 9, we can work by induction on n, with induction hypothesis

$$\mathcal{L}[t^ne^{at}] = \frac{n!}{(s-a)^{n+1}}.$$

Alternately, we can compute

$$\mathcal{L}[t^ne^{at}] = \int_0^\infty t^ne^{at}e^{-st}\,dt = \int_0^\infty t^ne^{-(s-a)t}\,dt = \int_0^\infty t^ne^{-rt}\,dt$$

where $r = s - a$. Now the last integral is the Laplace transform of t^n using r as the independent variable, so

$$\int_0^\infty t^ne^{-rt}\,dt = \frac{n!}{r^{n+1}}$$

from the table. Hence, substituting $r = s - a$ we have

$$\mathcal{L}[t^ne^{at}] = \frac{n!}{(s-a)^{n+1}}.$$

11. In this case, $b = 2$, and $(s + b/2)^2 = (s+1)^2 = s^2 + 2s + 1$, so $s^2 + 2s + 10 = (s+1)^2 + 3^2$.

12. In this case, $b = -4$, and $(s + b/2)^2 = (s-2)^2 = s^2 - 4s + 4$, so $s^2 - 4s + 5 = (s-2)^2 + 1^2$.

13. In this case, $b = 1$, and $(s + b/2)^2 = (s+1/2)^2 = s^2 + s + 1/4$, so $s^2 + s + 1 = (s+1/2)^2 + 3/4 = (s+1/2)^2 + (\sqrt{3}/2)^2$.

14. In this case, $b = 6$, and $(s + b/2)^2 = (s+3)^2 = s^2 + 6s + 9$, so $s^2 + 6s + 10 = (s+3)^2 + 1^2$.

15. In Exercise 11, we completed the square and obtained $s^2 + 2s + 10 = (s+1)^2 + 3^2$, so

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{s^2+2s+10}\right] &= \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+3^2}\right] \\ &= \frac{1}{3}\mathcal{L}^{-1}\left[\frac{3}{(s+1)^2+3^2}\right] \\ &= \frac{1}{3}e^{-t}\sin 3t.\end{aligned}$$

16. In Exercise 12, we completed the square and obtained $s^2 - 4s + 5 = (s-2)^2 + 1^2$, so

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{s}{s^2-4s+5}\right] &= \mathcal{L}^{-1}\left[\frac{s}{(s-2)^2+1^2}\right] \\ &= \mathcal{L}^{-1}\left[\frac{s-2}{(s-2)^2+1^2}\right] + \mathcal{L}^{-1}\left[\frac{2}{(s-2)^2+1^2}\right] \\ &= e^{2t}\cos t + e^{2t}(2\sin t) = e^{2t}(\cos t + 2\sin t).\end{aligned}$$

17. In Exercise 13, we completed the square and obtained $s^2 + s + 1 = (s+1/2)^2 + (\sqrt{3}/2)^2$, so

$$\frac{2s+3}{s^2+s+1} = \frac{2s+3}{(s+1/2)^2+(\sqrt{3}/2)^2}.$$

We want to put this fraction in the right form so that we can use the formulas for $\mathcal{L}[e^{at}\cos\omega t]$ and $\mathcal{L}[e^{at}\sin\omega t]$. We see that

$$\begin{aligned}\frac{2s+3}{(s+1/2)^2+(\sqrt{3}/2)^2} &= \frac{2s+1}{(s+1/2)^2+(\sqrt{3}/2)^2} + \frac{2}{(s+1/2)^2+(\sqrt{3}/2)^2} \\ &= \frac{2(s+1/2)}{(s+1/2)^2+(\sqrt{3}/2)^2} + \frac{(4/\sqrt{3})(\sqrt{3}/2)}{(s+1/2)^2+(\sqrt{3}/2)^2}.\end{aligned}$$

So

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{2s+3}{s^2+s+1}\right] &= 2\mathcal{L}^{-1}\left[\frac{(s+1/2)}{(s+1/2)^2+(\sqrt{3}/2)^2}\right] + \frac{4}{\sqrt{3}}\mathcal{L}^{-1}\left[\frac{\sqrt{3}/2}{(s+1/2)^2+(\sqrt{3}/2)^2}\right] \\ &= 2e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{4}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right).\end{aligned}$$

8. In Exercise 14, we completed the square and obtained $s^2 + 6s + 10 = (s + 3)^2 + 1^2$, so

$$\frac{s+1}{s^2+6s+10} = \frac{s+1}{(s+3)^2+1^2}.$$

We want to put this fraction in the right form so that we can use the formulas for $\mathcal{L}[e^{at}\cos\omega t]$ and $\mathcal{L}[e^{at}\sin\omega t]$. We see that

$$\frac{s+1}{(s+3)^2+1^2} = \frac{s+3}{(s+3)^2+1^2} - \frac{2}{(s+3)^2+1^2}.$$

So

$$\mathcal{L}^{-1}\left[\frac{s+1}{s^2+6s+10}\right] = e^{-3t}\cos t - 2e^{-3t}\sin t.$$

9. We compute

$$\begin{aligned}\mathcal{L}\left[e^{(a+ib)t}\right] &= \int_0^\infty e^{(a+ib)t}e^{-st}\,dt \\ &= \int_0^\infty e^{-(s-(a+ib))t}\,dt \\ &= -\frac{1}{s-(a+ib)}\left(\lim_{u\to\infty}\left[e^{-(s-a)u}e^{-ibu}\right] - 1\right).\end{aligned}$$

The limit is zero as long as $s > a$. Hence,

$$\mathcal{L}\left[e^{(a+ib)t}\right] = \frac{1}{s-(a+ib)}$$

if $s > a$ and undefined otherwise. This is the same formula as for real exponentials. It can also be written

$$\mathcal{L}\left[e^{(a+ib)t}\right] = \frac{s-a+ib}{(s-a)^2+b^2}.$$

20. This follows from linearity:

$$\begin{aligned}\mathcal{L}[y] &= \mathcal{L}[y_{\text{re}} + iy_{\text{im}}] \\ &= \int_0^\infty (y_{\text{re}} + iy_{\text{im}})\,e^{-st}\,dt \\ &= \int_0^\infty y_{\text{re}}(t)\,e^{-st}\,dt + i\int_0^\infty y_{\text{im}}(t)\,e^{-st}\,dt \\ &= \mathcal{L}[y_{\text{re}}] + i\mathcal{L}[y_{\text{im}}].\end{aligned}$$

Taking inverse Laplace transforms of the right-hand side gives

$$\left(1 - \tfrac{2}{\sqrt{3}}i\right) e^{(-1+i\sqrt{3})t/2} + \left(1 + \tfrac{2}{\sqrt{3}}i\right) e^{(-1-i\sqrt{3})t/2}.$$

Using Euler's formula to replace the complex exponentials and simplifying yields

$$2e^{-t/2} \cos\left(\tfrac{\sqrt{3}}{2}t\right) + \tfrac{4}{\sqrt{3}}e^{-t/2} \sin\left(\tfrac{\sqrt{3}}{2}t\right).$$

26. Using the quadratic formula, we find the roots of the denominator are $-3 \pm i$ so the denominator can be factored

$$s^2 + 6s + 10 = (s - (-3 + i))(s - (-3 - i)).$$

The partial fractions decomposition is

$$\frac{s+1}{s^2 + 6s + 10} = \frac{A}{s - (-3 - i)} + \frac{B}{s - (-3 + i)},$$

which leads to the equations

$$\begin{cases} A + B = 1 \\ (3 - i)A + (3 + i)B = 1. \end{cases}$$

Solving, we find $A = \frac{1}{2} - i$ and $B = \frac{1}{2} + i$, so

$$\frac{s+1}{s^2 + 6s + 10} = \frac{\frac{1}{2} - i}{s - (-3 - i)} + \frac{\frac{1}{2} + i}{s - (-3 + i)}.$$

Taking inverse Laplace transform of the right-hand side gives

$$\left(\tfrac{1}{2} - i\right) e^{(-3-i)t} + \left(\tfrac{1}{2} + i\right) e^{(-3+i)t}$$

and using Euler's formula and simplifying gives

$$e^{-3t} \cos t - 2e^{-3t} \sin t.$$

27. **(a)** Taking the Laplace transform of both sides of the equation, we obtain

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] + 4\mathcal{L}[y] = \frac{8}{s},$$

and using the fact that $\mathcal{L}[d^2y/dt^2] = s^2\mathcal{L}[y] - sy(0) - y'(0)$, we have

$$(s^2 + 4)\mathcal{L}[y] - sy(0) - y'(0) = \frac{8}{s}.$$

(b) Substituting the initial conditions yields

$$(s^2+4)\mathcal{L}[y] - 11s - 5 = \frac{8}{s},$$

and solving for $\mathcal{L}[y]$ we get

$$\mathcal{L}[y] = \frac{11s+5}{s^2+4} + \frac{8}{s(s^2+4)}.$$

The partial fractions decomposition of $8/(s(s^2+4))$ is

$$\frac{8}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}.$$

Putting the right-hand side over a common denominator gives us

$$(A+B)s^2 + Cs + 4A = 8,$$

and consequently, $A=2$, $B=-2$, and $C=0$. In other words,

$$\frac{8}{s(s^2+4)} = \frac{2}{s} + \frac{-2s}{s^2+4}.$$

We obtain

$$\mathcal{L}[y] = \frac{2}{s} + \frac{9s+5}{s^2+4}.$$

(c) To take the inverse Laplace transform, we rewrite $\mathcal{L}[y]$ in the form

$$\mathcal{L}[y] = \frac{2}{s} + 9\left(\frac{s}{s^2+4}\right) + \frac{5}{2}\left(\frac{2}{s^2+4}\right).$$

Therefore, $y(t) = 2 + 9\cos 2t + \frac{5}{2}\sin 2t$.

28. **(a)** Taking the Laplace transform of both sides of the equation, we obtain

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] - \mathcal{L}[y] = \frac{1}{s-2},$$

and using the fact that $\mathcal{L}[d^2y/dt^2] = s^2\mathcal{L}[y] - sy(0) - y'(0)$, we have

$$(s^2-1)\mathcal{L}[y] - sy(0) - y'(0) = \frac{1}{s-2}.$$

(b) Substituting the initial conditions yields

$$(s^2-1)\mathcal{L}[y] - s + 1 = \frac{1}{s-2},$$

and solving for $\mathcal{L}[y]$ we get

$$\mathcal{L}[y] = \frac{1}{s+1} + \frac{1}{(s-2)(s^2-1)}.$$

Using the partial fractions decomposition

$$\frac{1}{(s-2)(s^2-1)} = \frac{\frac{1}{3}}{s-2} + \frac{-\frac{1}{2}}{s-1} + \frac{\frac{1}{6}}{s+1},$$

we obtain

$$\mathcal{L}[y] = \frac{\frac{1}{3}}{s-2} + \frac{-\frac{1}{2}}{s-1} + \frac{\frac{7}{6}}{s+1}.$$

(c) Taking the inverse Laplace transform, we have

$$y(t) = \tfrac{1}{3}e^{2t} - \tfrac{1}{2}e^{t} + \tfrac{7}{6}e^{-t}.$$

29. **(a)** Taking the Laplace transform of both sides of the equation, we obtain

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] - 4\mathcal{L}\left[\frac{dy}{dt}\right] + 5\mathcal{L}[y] = \frac{2}{s-1},$$

and using the formulas for $\mathcal{L}[dy/dt]$ and $\mathcal{L}[d^2y/dt^2]$ in terms of $\mathcal{L}[y]$, we have

$$(s^2 - 4s + 5)\mathcal{L}[y] - sy(0) - y'(0) + 4y(0) = \frac{2}{s-1}.$$

(b) Substituting the initial conditions yields

$$(s^2 - 4s + 5)\mathcal{L}[y] - 3s + 11 = \frac{2}{s-2},$$

and solving for $\mathcal{L}[y]$ we get

$$\mathcal{L}[y] = \frac{3s-11}{s^2-4s+5} + \frac{2}{(s-1)(s^2-4s+5)}.$$

Using the partial fractions decomposition

$$\frac{2}{(s-1)(s^2-4s+5)} = \frac{1}{s-1} + \frac{-s+3}{s^2-4s+5},$$

we obtain

$$\mathcal{L}[y] = \frac{1}{s-1} + \frac{2s-8}{s^2-4s+5}.$$

(c) In order to compute the inverse Laplace transform, we first write

$$s^2 - 4s + 5 = (s-2)^2 + 1$$

by completing the square, and then we write

$$\frac{2s-8}{s^2-4s+5} = \frac{2(s-2)}{(s-2)^2+1} - \frac{4}{(s-2)^2+1}.$$

Taking the inverse Laplace transform, we have

$$y(t) = e^{t} + 2e^{2t}\cos t - 4e^{2t}\sin t.$$