EXERCISES FOR SECTION 6.1

1. We have

\[ \mathcal{L}[3] = \int_0^\infty 3e^{-st} \, dt \]

\[ = \lim_{b \to \infty} \int_0^b 3e^{-st} \, dt \]

\[ = \lim_{b \to \infty} \left( \frac{-3}{s} e^{-st} \bigg|_0^b \right) \]

\[ = \lim_{b \to \infty} \frac{3}{s} \left( e^{-sb} - e^0 \right) \]

\[ = \frac{3}{s} \text{ if } s > 0, \]

since \( \lim_{b \to \infty} e^{-sb} = \lim_{b \to \infty} 1/e^{sb} = 0 \) if \( s > 0 \).

2. We have

\[ \mathcal{L}[t] = \int_0^\infty te^{-st} \, dt = \lim_{b \to \infty} \int_0^b te^{-st} \, dt. \]

To evaluate the integral we use integration by parts with \( u = t \) and \( dv = e^{-st} \, dt \). Then \( du = dt \) and \( v = -e^{-st}/s \). Thus

\[ \lim_{b \to \infty} \int_0^b te^{-st} \, dt = \lim_{b \to \infty} \left( \frac{-te^{-st}}{s} \bigg|_0^b - \int_0^b \frac{-e^{-st}}{s} \, dt \right) \]

\[ = \lim_{b \to \infty} \left( \frac{-be^{-sb}}{s} - \frac{e^{-st}}{s^2} \bigg|_0^b \right) \]

\[ = \lim_{b \to \infty} \left( -\frac{be^{-sb}}{s} - \frac{e^{-sb}}{s^2} + \frac{e^0}{s^2} \right) \]

\[ = \frac{1}{s^2} \]

since

\[ \lim_{b \to \infty} -\frac{be^{-sb}}{s} = \lim_{b \to \infty} -\frac{b}{se^{sb}} = \lim_{b \to \infty} \frac{-1}{s^2 e^{sb}} = 0 \]

by L'Hôpital's Rule if \( s > 0 \).

3. We use the fact that \( \mathcal{L}[df/dt] = s \mathcal{L}[f] - f(0) \). Letting \( f(t) = t^2 \) we have \( f(0) = 0 \) and

\[ \mathcal{L}[2t] = s \mathcal{L}[t^2] - 0 \]

or

\[ 2 \mathcal{L}[t] = s \mathcal{L}[t^2] \]
using the fact that the Laplace transform is linear. Then since $\mathcal{L}[t] = 1/s^2$ (by the previous exercise), we have

$$\mathcal{L}[-5t^2] = -5\mathcal{L}[t^2] = -5 \left( \frac{2\mathcal{L}[t]}{s} \right) = -\frac{10}{s^3}.$$ 

4. We have shown thus far that $\mathcal{L}[t] = 1/s^2$ and $\mathcal{L}[t^2] = 2/s^3$. Let’s compute $\mathcal{L}[t^3]$ and see if a pattern emerges. Using $\mathcal{L}[df/dt] = s\mathcal{L}[f] - f(0)$ with $f(t) = t^3$, we have

$$\mathcal{L}[3t^2] = 3\mathcal{L}[t^2] = s\mathcal{L}[t^3] - f(0)$$

which yields

$$\mathcal{L}[t^3] = \frac{3}{s}\mathcal{L}[t^2] = \frac{3 \cdot 2}{s^4} = \frac{3!}{s^4}.$$ 

If we were to continue, we would see that

$$\mathcal{L}[t^4] = \frac{4}{s}\mathcal{L}[t^3] = \frac{4!}{s^5}.$$ 

A clear pattern has emerged (which we could prove by induction should we be so inclined):

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

so that $\mathcal{L}[t^5] = 5!/s^6$.

5. To show a rule by induction, we need two steps. First, we need to show the rule is true for $n = 1$. Then, we need to show that if the rule holds for $n$, then it holds for $n + 1$.

(a) $n = 1$. We need to show that $\mathcal{L}[t] = 1/s^2$.

$$\mathcal{L}[t] = \int_0^{\infty} te^{-st} \, dt.$$ 

Using integration by parts with $u = t$ and $dv = e^{-st} \, dt$, we find

$$\mathcal{L}[t] = \left[ \frac{te^{-st}}{-s} \right]_0^\infty + \int_0^{\infty} \frac{e^{-st}}{s} \, dt$$

$$= \lim_{b \to \infty} \left[ \frac{te^{-st}}{-s} \right]_0^b + \int_0^{\infty} \frac{e^{-st}}{s} \, dt$$

$$= \int_0^{\infty} \frac{e^{-st}}{s} \, dt$$

$$= -\frac{e^{-st}}{s^2} \bigg|_0^{\infty}$$

$$= \frac{1}{s^2} \quad (s > 0).$$

(b) Now we assume that the rule holds for $n$, that is, that $\mathcal{L}[t^n] = n!/s^{n+1}$, and show it holds true for $n + 1$, that is, $\mathcal{L}[t^{n+1}] = (n + 1)!/s^{n+2}$. There are two different methods to do so:

(i) 

$$\mathcal{L}[t^{n+1}] = \int_0^{\infty} t^{n+1} e^{-st} \, dt$$
Using integration by parts with \( u = t^{n+1} \) and \( dv = e^{-st} \, dt \), we find
\[
\mathcal{L}[t^{n+1}] = \lim_{b \to \infty} \left[ -\frac{t^{n+1}e^{-st}}{s} \right]_0^b + \int_0^\infty \frac{n+1}{s} t^n e^{-st} \, dt.
\]
Now,
\[
-\frac{t^{n+1}e^{-st}}{s} \bigg|_0^b = \lim_{b \to \infty} \left[ -\frac{t^{n+1}e^{-st}}{s} \right]_0^b
= \lim_{b \to \infty} \frac{-b^{n+1}e^{-sb}}{s} + 0
= 0 \quad (s > 0).
\]
So
\[
\mathcal{L}[t^{n+1}] = \int_0^\infty \frac{n+1}{s} t^n e^{-st} \, dt
= \frac{n+1}{s} \int_0^\infty t^n e^{-st} \, dt
= \frac{n+1}{s} \mathcal{L}[t^n].
\]
Since we assumed that \( \mathcal{L}[t^n] = n! / s^{n+1} \), we get that
\[
\mathcal{L}[t^{n+1}] = \frac{n+1}{s} \cdot \frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}}
\]
which is what we wanted to show.

(ii) We use the fact that \( \mathcal{L}[df/dt] = s \mathcal{L}[f] - f(0) \). Letting \( f(t) = t^{n+1} \) we have \( f(0) = 0 \) and
\[
\mathcal{L}[(n+1)t^n] = s \mathcal{L}[t^{n+1}] - 0
\]
or
\[
(n+1) \mathcal{L}[t^n] = s \mathcal{L}[t^{n+1}]
\]
using the fact that the Laplace transform is linear. Since we assumed \( \mathcal{L}[t^n] = n! / s^{n+1} \), we have
\[
\mathcal{L}[t^{n+1}] = \frac{n+1}{s} \mathcal{L}[t^n] = \frac{n+1}{s} \cdot \frac{n!}{s^{n+1}} = \frac{(n+1)!}{s^{n+2}}
\]
which is what we wanted to show.

6. Using the formula that \( \mathcal{L}[t^n] = n! / s^{n+1} \), we can see by linearity that
\[
\mathcal{L}[a_i t^i] = a_i \frac{i!}{s^{i+1}}
\]
so
\[
\mathcal{L}[a_0 + a_1 t + \cdots + a_n t^n] = \mathcal{L}[a_0] + \mathcal{L}[a_1 t] + \cdots + \mathcal{L}[a_n t^n]
= \frac{a_0}{s} + \frac{a_1}{s^2} + \cdots + \frac{a_n n!}{s^{n+1}}.
\]
7. Since we know that $\mathcal{L}[e^{at}] = 1/(s-a)$, we have $\mathcal{L}[e^{3t}] = 1/(s-3)$, and therefore,

$$\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] = e^{3t}.$$ 

8. We see that

$$\frac{5}{3s} = \frac{5}{3} \cdot \frac{1}{s},$$

so

$$\mathcal{L}^{-1}\left[\frac{5}{3s}\right] = \frac{5}{3},$$

since $\mathcal{L}^{-1}[1/s] = 1$.

9. We see that

$$\frac{2}{3s + 5} = \frac{2}{3} \cdot \frac{1}{s + 5/3},$$

so

$$\mathcal{L}^{-1}\left[\frac{2}{3s + 5}\right] = \frac{2}{3}e^{-5t/3}.$$

10. Using the method of partial fractions,

$$\frac{14}{(3s + 2)(s - 4)} = \frac{A}{3s + 2} + \frac{B}{s - 4}.$$ 

Putting the right-hand side over a common denominator gives $A(s - 4) + B(3s + 2) = 14$, which can be written as $(A + 3B)s + (-4A + 2B) = 14$. So, $A + 3B = 0$ and $-4A + 2B = 14$. Thus $A = -3$ and $B = 1$, and

$$\mathcal{L}^{-1}\left[\frac{14}{(3s + 2)(s - 4)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s - 4} - \frac{3}{3s + 2}\right].$$

Finally,

$$\mathcal{L}^{-1}\left[\frac{5}{(s - 1)(s - 2)}\right] = e^{4t} - e^{-2t/3}.$$ 

Using the method of partial fractions, we write

$$\frac{4}{s(s + 3)} = \frac{A}{s} + \frac{B}{s + 3}.$$ 

Putting the right-hand side over a common denominator gives $A(s + 3) + B(s) = 4$, which can be written as $(A + B)s + 3A = 4$. Thus, $A + B = 0$, and $3A = 4$. This gives $A = 4/3$ and $B = -4/3$, so

$$\mathcal{L}^{-1}\left[\frac{4}{s(s + 3)}\right] = \mathcal{L}^{-1}\left[\frac{4/3}{s} - \frac{4/3}{s + 3}\right].$$

Hence,

$$\mathcal{L}^{-1}\left[\frac{4}{s(s + 3)}\right] = \frac{4}{3} - \frac{4}{3}e^{-3t}.$$
12. Using the method of partial fractions, we write

\[
\frac{5}{(s - 1)(s - 2)} = \frac{A}{s - 1} + \frac{B}{s - 2}.
\]

Putting the right-hand side over a common denominator gives \(A(s - 2) + B(s - 1) = 5\), which can be written as \((A + B)s + (-2A - B) = 5\). Thus, \(A + B = 0\), and \(-2A - B = 5\). This gives \(A = -5\) and \(B = 5\), so

\[
\mathcal{L}^{-1}\left[\frac{5}{(s - 1)(s - 2)}\right] = \mathcal{L}^{-1}\left[\frac{5}{s - 2} - \frac{5}{s - 1}\right].
\]

Thus,

\[
\mathcal{L}^{-1}\left[\frac{5}{(s - 1)(s - 2)}\right] = 5e^{2t} - 5e^t.
\]

13. Using the method of partial fractions, we have

\[
\frac{2s + 1}{(s - 1)(s - 2)} = \frac{A}{s - 1} + \frac{B}{s - 2}.
\]

Putting the right-hand side over a common denominator gives \(A(s - 2) + B(s - 1) = 2s + 1\), which can be written as \((A + B)s + (-2A - B) = 2s + 1\). So, \(A + B = 2\), and \(-2A - B = 1\). Thus \(A = -3\) and \(B = 5\), which gives

\[
\mathcal{L}^{-1}\left[\frac{2s + 1}{(s - 1)(s - 2)}\right] = \mathcal{L}^{-1}\left[\frac{5}{s - 2} - \frac{3}{s - 1}\right].
\]

Finally,

\[
\mathcal{L}^{-1}\left[\frac{2s + 1}{(s - 1)(s - 2)}\right] = 5e^{2t} - 3e^t.
\]

14. Using the method of partial fractions,

\[
\frac{2s^2 + 3s - 2}{s(s + 1)(s - 2)} = \frac{A}{s} + \frac{B}{s + 1} + \frac{C}{s - 2}.
\]

Putting the right-hand side over a common denominator gives

\[
A(s + 1)(s - 2) + Bs(s - 2) + Cs(s + 1) = 2s^2 + 3s - 2,
\]

which can be written as \((A + B + C)s^2 + (-A - 2B + C)s - 2A = 2s^2 + 3s - 2\). So, \(A + B + C = 2\), \(-A - 2B + C = 3\), and \(-2A = -2\). Thus \(A = 1\), \(B = -1\), and \(C = 2\), and

\[
\mathcal{L}^{-1}\left[\frac{2s^2 + 3s - 2}{s(s + 1)(s - 2)}\right] = \mathcal{L}^{-1}\left[\frac{2}{s - 2} - \frac{1}{s + 1} + \frac{1}{s}\right].
\]

Hence,

\[
\mathcal{L}^{-1}\left[\frac{2s^2 + 3s - 2}{s(s + 1)(s - 2)}\right] = 2e^{2t} - e^{-t} + 1.
\]
15. (a) We have
\[ \mathcal{L}\left[ \frac{dy}{dt} \right] = s \mathcal{L}[y] - y(0) \]
and
\[ \mathcal{L}[-y + e^{-2t}] = \mathcal{L}[-y] + \mathcal{L}[e^{-2t}] = -\mathcal{L}[y] + \frac{1}{s+2} \]
using linearity of the Laplace transform and the formula \( \mathcal{L}[e^{at}] = 1/(s-a) \) from the text.
(b) Substituting the initial condition yields
\[ s \mathcal{L}[y] - 2 = -\mathcal{L}[y] + \frac{1}{s+2} \]
so that
\[ (s+1) \mathcal{L}[y] = 2 + \frac{1}{s+2} \]
which gives
\[ \mathcal{L}[y] = \frac{1}{(s+1)(s+2)} + \frac{2}{s+1} = \frac{2s+5}{(s+1)(s+2)}. \]
(c) Using the method of partial fractions,
\[ \frac{2s+5}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}. \]
Putting the right-hand side over a common denominator gives \( A(s+2) + B(s+1) = 2s + 5 \),
which can be written as \( (A+B)s + (2A+B) = 2s + 5 \). So we have \( A+B = 2 \), and \( 2A+B = 5 \).
Thus, \( A = 3 \) and \( B = -1 \), and
\[ \mathcal{L}[y] = \frac{3}{s+1} - \frac{1}{s+2}. \]
Therefore, \( y(t) = 3e^{-t} - e^{-2t} \) is the desired function.
(d) Since \( y(0) = 3e^0 - e^0 = 2 \), \( y(t) \) satisfies the given initial condition. Also,
\[ \frac{dy}{dt} = -3e^{-t} + 2e^{-2t} \]
and
\[ -y + e^{-2t} = -3e^{-t} + e^{-2t} + e^{-2t} = -3e^{-t} + 2e^{-2t}, \]
so our solution also satisfies the differential equation.

16. (a) Taking Laplace transforms of both sides of the equation and simplifying gives
\[ \mathcal{L}\left[ \frac{dy}{dt} \right] + 5 \mathcal{L}[y] = \mathcal{L}[e^{-t}] \]
so
\[ s \mathcal{L}[y] - y(0) + 5 \mathcal{L}[y] = \frac{1}{s+1} \]
and \( y(0) = 2 \) gives
\[ s \mathcal{L}[y] - 2 + 5 \mathcal{L}[y] = \frac{1}{s+1}. \]
(b) Solving for $\mathcal{L}[y]$ gives

$$\mathcal{L}[y] = \frac{2}{s+5} + \frac{1}{(s+5)(s+1)} = \frac{2s+3}{(s+5)(s+1)}.$$ 

(c) Using the method of partial fractions,

$$\frac{2s+3}{(s+5)(s+1)} = \frac{A}{s+5} + \frac{B}{s+1}.$$ 

Putting the right-hand side over a common denominator gives $A(s+1) + B(s+5) = 2s + 3$, which can be written as $(A + B)s + (A + 5B) = 2s + 3$. So $A + B = 2$, and $A + 5B = 3$. Hence, $A = 7/4$ and $B = 1/4$ and we have

$$\mathcal{L}[y] = \frac{7/4}{s+5} + \frac{1/4}{s+1}.$$ 

Therefore,

$$y(t) = \frac{7}{4}e^{-5t} + \frac{1}{4}e^{-t}.$$ 

(d) To check, compute

$$\frac{dy}{dt} + 5y = \frac{-35}{4}e^{-5t} - \frac{1}{4}e^{-t} + 5\left(\frac{7}{4}e^{-5t} + \frac{1}{4}e^{-t}\right) = e^{-t},$$

and $y(0) = 7/4 + 1/4 = 2$.

17. (a) Taking Laplace transforms of both sides of the equation and simplifying gives

$$\mathcal{L}\left[\frac{dy}{dt}\right] + 7\mathcal{L}[y] = \mathcal{L}[1]$$

so

$$s\mathcal{L}[y] - y(0) + 7\mathcal{L}[y] = \frac{1}{s}$$

and $y(0) = 3$ gives

$$s\mathcal{L}[y] - 3 + 7\mathcal{L}[y] = \frac{1}{s}.$$ 

(b) Solving for $\mathcal{L}[y]$ gives

$$\mathcal{L}[y] = \frac{3}{s+7} + \frac{1}{s(s+7)} = \frac{3s+1}{s(s+7)}.$$ 

(c) Using the method of partial fractions, we get

$$\frac{3s+1}{s(s+7)} = \frac{A}{s} + \frac{B}{s+7}.$$ 

Putting the right-hand side over a common denominator gives $A(s+7) + Bs = 3s + 1$, which can be written as $(A + B)s + 7A = 3s + 1$. So $A + B = 3$, and $7A = 1$. Hence, $A = 1/7$ and $B = 20/7$, and we have

$$\mathcal{L}[y] = \frac{1/7}{s} + \frac{20/7}{s+7}.$$ 

Thus,

$$y(t) = \frac{20}{7}e^{-7t} + \frac{1}{7}. $$
(d) To check, we compute

\[ \frac{dy}{dt} + 7y = -20e^{-7t} + 7 \left( \frac{20}{7} e^{-7t} + \frac{1}{7} \right) = 1, \]

and \( y(0) = 20/7 + 1/7 = 3 \), so our solution satisfies the initial-value problem.

18. (a) Taking Laplace transforms of both sides of the equation and simplifying gives

\[ \mathcal{L} \left[ \frac{dy}{dt} \right] + 4 \mathcal{L}[y] = \mathcal{L}[6] \]

so

\[ s \mathcal{L}[y] - y(0) + 4 \mathcal{L}[y] = \frac{6}{s} \]

and \( y(0) = 0 \) gives

\[ s \mathcal{L}[y] + 4 \mathcal{L}[y] = \frac{6}{s}. \]

(b) Solving for \( \mathcal{L}[y] \) gives

\[ \mathcal{L}[y] = \frac{6}{s(s + 4)}. \]

(c) Using the method of partial fractions,

\[ \frac{6}{s(s + 4)} = \frac{A}{s} + \frac{B}{s + 4}. \]

Putting the right-hand side over a common denominator gives \( A(s + 4) + Bs = 6 \), which can be written as \( (A + B)s + 4A = 6 \). So, \( A + B = 0 \), and \( 4A = 6 \). Hence, \( A = 3/2 \) and \( B = -3/2 \), and we have

\[ \mathcal{L}[y] = \frac{3/2}{s} - \frac{3/2}{s + 4}. \]

Thus,

\[ y(t) = \frac{3}{2} - \frac{3}{2} e^{-4t}. \]

(d) To check, we compute

\[ \frac{dy}{dt} + 4y = 6e^{-4t} + 4 \left( \frac{3}{2} - \frac{3}{2} e^{-4t} \right) = 6, \]

and \( y(0) = 3/2 - 3/2 = 0 \), so our solution satisfies the initial-value problem.

19. (a) Taking Laplace transforms of both sides of the equation and simplifying gives

\[ \mathcal{L} \left[ \frac{dy}{dt} \right] + 9 \mathcal{L}[y] = \mathcal{L}[2] \]

so

\[ s \mathcal{L}[y] - y(0) + 9 \mathcal{L}[y] = \frac{2}{s} \]

and \( y(0) = -2 \) gives

\[ s \mathcal{L}[y] + 2 + 9 \mathcal{L}[y] = \frac{2}{s}. \]