2. (a) If \( H(x, y) = \sin(xy) \), then

\[
\frac{\partial H}{\partial x} = y \cos(xy)
\]

and so

\[
\frac{dy}{dt} = -\frac{\partial H}{\partial x}.
\]

Similarly,

\[
\frac{\partial H}{\partial y} = x \cos(xy) = \frac{dx}{dt}.
\]

(b) Note that the level sets of \( H \) are the same curves as those of the level sets of \( xy \).

(c) Note that there are many curves of equilibrium points for this system: besides the origin, whenever \( xy = n\pi + \pi/2 \), the vector field vanishes.

3. (a) If \( H(x, y) = x \cos y + y^2 \), then

\[
\frac{\partial H}{\partial x} = \cos y
\]

and so

\[
\frac{dy}{dt} = -\frac{\partial H}{\partial x}.
\]

Similarly,

\[
\frac{\partial H}{\partial y} = -x \sin y + 2y = \frac{dx}{dt}.
\]
If we differentiate $H(x, y)$ with respect to $x$, we get

$$y + c'(x),$$

which we want to be the negative of $dy/dt = -y$. Hence $c'(x) = 0$, and we pick the antiderivative $c(x) = 0$. A Hamiltonian function is

$$H(x, y) = xy - y^3.$$

12. First we check to see if the partial derivative with respect to $x$ of the first component of the vector field is the negative of the partial derivative with respect to $y$ of the second component. We have

$$\frac{\partial 1}{\partial x} = 0$$

while

$$\frac{\partial y}{\partial y} = -1.$$

Since these are not equal, the system is not Hamiltonian.

13. First we check to see if the partial derivative with respect to $x$ of the first component of the vector field is the negative of the partial derivative with respect to $y$ of the second component. We have

$$\frac{\partial (x \cos y)}{\partial x} = \cos y$$

while

$$-\frac{\partial (-y \cos x)}{\partial y} = \cos x.$$

Since these two are not equal, the system is not Hamiltonian.

14. First note that

$$\frac{\partial F(y)}{\partial x} = 0 = -\frac{\partial G(x)}{\partial y},$$

that is, the partial derivative of the $x$ component of the vector field with respect to $x$ is equal to the negative of the partial derivative of the $y$ component with respect to $y$. Hence, the system is Hamiltonian. Integrating the $x$ component of the vector field with respect to $y$ yields

$$H(x, y) = \int F(y) \, dy + c$$

where the "constant" $c$ could depend on $x$. If we differentiate this $H$ with respect to $x$ we get

$$-\frac{\partial H}{\partial x} = -c'(x).$$

Thus we take $c = -\int G(x) \, dx$. A Hamiltonian function is

$$H(x, y) = \int F(y) \, dy - \int G(x) \, dx.$$
16. (a) We first check
\[
\frac{\partial (-yx^2)}{\partial x} = -2xy \neq -\frac{\partial (x + 1)}{\partial y} = 0,
\]
so the system is not Hamiltonian.
(b) If we multiply the vector field by \(1/x^2\), we obtain the new system
\[
\begin{align*}
\frac{dx}{dt} &= -y \\
\frac{dy}{dt} &= \frac{1}{x} + \frac{1}{x^2}.
\end{align*}
\]
As in Exercise 14, this system is Hamiltonian with
\[
H(x, y) = \frac{1}{x} - \ln |x| - \frac{y^2}{2}.
\]

17. Using the technique of Exercise 15, we we multiply the vector field by \(1/(2 - y)\). As in Exercise 14, the resulting system
\[
\begin{align*}
\frac{dx}{dt} &= \frac{1 - y^2}{2 - y} \\
\frac{dy}{dt} &= x
\end{align*}
\]
is Hamiltonian. The Hamiltonian is
\[
\begin{align*}
H(x, y) &= -\frac{x^2}{2} + \int \frac{y^2 - 1}{y - 2} dy \\
&= -\frac{x^2}{2} + \int 2 + y + \frac{3}{y - 2} dy \\
&= -\frac{x^2}{2} + 2y + \frac{y^2}{2} + 3\ln |y - 2|.
\end{align*}
\]
The function
\[
H(x, y) = -\frac{x^2}{2} + 2y + \frac{y^2}{2} + 3\ln |y - 2|
\]
is a conserved quantity for the original system. However, it is not defined on the line \(y = 2\). From the system, we see that this line is a single solution curve that separates the two half-planes, \(y < 2\) and \(y > 2\).

18. (a) We have
\[
\frac{\partial H}{\partial y} = y \quad \text{and} \quad \frac{\partial H}{\partial x} = x^2 - a,
\]
so this system is Hamiltonian with the given function \(H\).
(b) Note that \(dx/dt = 0\) if and only if \(y = 0\) and \(dy/dt = 0\) if and only if \(x = \pm \sqrt{a}\). Consequently if \(a < 0\), then there are no equilibrium points. If \(a = 0\), there is one equilibrium point at \((0, 0)\) and if \(a > 0\), there are two equilibrium points at \(\pm \sqrt{a}, 0\).
(c) The Jacobian matrix is
\[
\begin{pmatrix}
0 & 1 \\
2x & 0
\end{pmatrix},
\]
which, when evaluated at the equilibrium points, becomes
\[
\begin{pmatrix}
0 & 1 \\
\pm 2\sqrt{a} & 0
\end{pmatrix}.
\]
At \((\sqrt{a}, 0)\), the eigenvalues are \(\pm \sqrt{2\sqrt{a}}\) so this equilibrium point is a saddle. At \((-\sqrt{a}, 0)\), the eigenvalues are \(\pm i\sqrt{2\sqrt{a}}\) so this equilibrium point is a center. If \(a = 0\) the eigenvalues are both 0, so this point is a node.

(d) Phase portrait for \(a < 0\)

Phase portrait for \(a = 0\)

Phase portrait for \(a > 0\)

(e) As \(a\) increases toward 0, the phase portrait changes from having no equilibrium points to having a single equilibrium point at \(a = 0\). If \(a > 0\), there is a pair of equilibrium points.

19. First note that this system is Hamiltonian for every value of \(a\). The Hamiltonian function depends on \(a\) and is given by
\[H(x, y) = x^2y + xy^2 - ax.\]
If \(a > 0\), then the system has two saddle equilibrium points on the \(y\)-axis at \((0, \pm \sqrt{a})\). If \(a = 0\), then system has only one equilibrium point at \((0, 0)\). If \(a < 0\), the system again has two saddles, but they are now located at \((\pm 2\sqrt{-3a/3}, \mp \sqrt{-3a/3})\). This corresponds to a change in shape of the graph of \(H\).

20. (a) First note that the system is still Hamiltonian, with Hamiltonian function
\[H(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3 + ax.\]
The equilibrium points are
\[
\left(\frac{1 \pm \sqrt{1 - 4a}}{2}, 0\right).
\]
Hence there are no equilibrium points if \(a > 1/4\); one equilibrium point if \(a = 1/4\); and two equilibrium points if \(a < 1/4\). A bifurcation occurs at \(a = 1/4\).

(b) The book would never have appeared. Wouldn't that have been awful?