EXERCISES FOR SECTION 5.1

1. The linearizations of systems (i) and (iii) are both

$$\frac{dx}{dt} = 2x + y$$
$$\frac{dy}{dt} = -y,$$

so these two systems have the same "local picture" near (0, 0). This system has eigenvalues 2 and −1; hence, (0, 0) is a saddle for these systems. System (ii) has linearization

$$\frac{dx}{dt} = 2x + y$$
$$\frac{dy}{dt} = y,$$

which has eigenvalues 2 and 1, hence, (0, 0) is a source for this system.

2. The linearizations of systems (ii) and (iii) are both equal to

$$\frac{dx}{dt} = -3x + y$$
$$\frac{dy}{dt} = 4x$$

so these two systems have the same "local picture" near (0, 0). These systems have eigenvalues −4 and 1, hence, (0, 0) is a saddle for these systems. System (i) has linearization

$$\frac{dx}{dt} = 3x + y$$
$$\frac{dy}{dt} = 4x$$

which has eigenvalues 4 and −1 so that (0, 0) is also a saddle for this system. However, the eigenvector corresponding to the eigenvalue −4 in systems (ii) and (iii) lie on the line $y = -x$, whereas the eigenvectors corresponding to the eigenvalue −1 for system (i) lie along the line $y = -4x$.

3. (a) The linearized system is

$$\frac{dx}{dt} = -2x + y$$
$$\frac{dy}{dt} = -y.$$

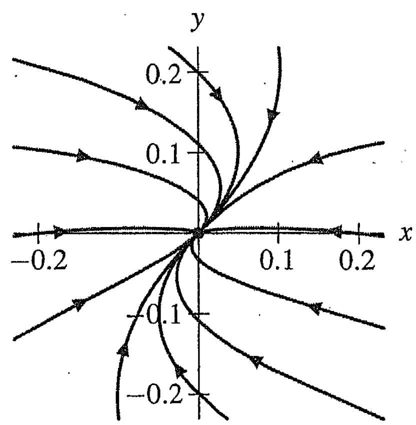
We can see this either by "dropping higher-order terms" or by computing the Jacobian matrix

$$\begin{pmatrix} -2 & 1 \\ 2x & -1 \end{pmatrix}$$

and evaluating it at (0, 0).

(b) The eigenvalues of the linearized system are -2 and -1, so $(0, 0)$ is a sink.

(c) The vector $(1, 0)$ is an eigenvector for eigenvalue -2 and $(1, 1)$ is an eigenvector for the eigenvalue -1.



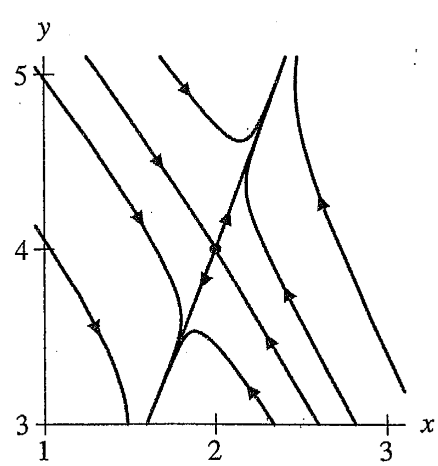
(d) By computing the Jacobian matrix

$$\begin{pmatrix} -2 & 1 \\ 2x & -1 \end{pmatrix}$$

and evaluating at $(2, 4)$, we see that linearized system at $(2, 4)$ is

$$\frac{dx}{dt} = -2x + y$$
$$\frac{dy}{dt} = 4x - y.$$

Its eigenvalues are $(-3 \pm \sqrt{17})/2$, so $(2, 4)$ is a saddle.



4. (a) The equilibrium points occur where the vector field is zero, that is, at solutions of

$$\begin{cases} -x = 0 \\ -4x^3 + y = 0. \end{cases}$$

So, $x = y = 0$ is the only equilibrium point.

(b) The Jacobian matrix of this system is

$$\begin{pmatrix} -1 & 0 \\ -12x^2 & 1 \end{pmatrix},$$

which at (0, 0) is equal to

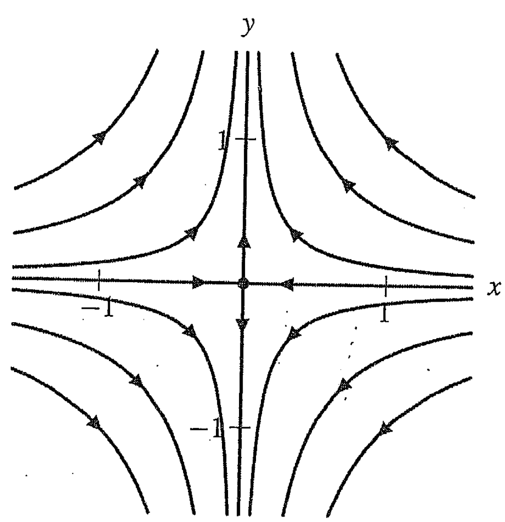
$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So the linearized system at (0, 0) is

$$\frac{dx}{dt} = -x$$
$$\frac{dy}{dt} = y$$

(we could also see this by "dropping the higher order terms").

(c) The eigenvalues of the linearized system at the origin are -1 and 1, so the origin is a saddle. The linearized system decouples, so solutions approach the origin along the x-axis and tend away form the origin along the y-axis.



5. (a) Using separation of variables (or simple guessing), we have $x(t) = x_0 e^{-t}$.

(b) Using the result in part (a), we can rewrite the equation for dy/dt as

$$\frac{dy}{dt} = y - 4x_0^3 e^{-3t}.$$

This first-order equation is a nonhomogeneous linear equation.

The general solution of its associated homogeneous equation is ke^t. To find a particular solution to the nonhomogeneous equation, we rewrite it as

$$\frac{dy}{dt} - y = -4x_0^3 e^{-3t},$$

and we guess a solution of the form $y_p = \alpha e^{-3t}$. Substituting this guess into the left-hand side of the equation yields

$$\frac{dy_p}{dt} - y_p = -4\alpha e^{-3t}.$$

Therefore, y_p is a solution if $\alpha = x_0^3$. The general solution of the original equation is

$$y(t) = x_0^3 e^{-3t} + ke^t.$$

To express this result in terms of the initial condition $y(0) = y_0$, we evaluate at $t = 0$ and note that $k = y_0 - x_0^3$. We conclude that

$$y(t) = x_0^3 e^{-3t} + (y_0 - x_0^3)e^t.$$

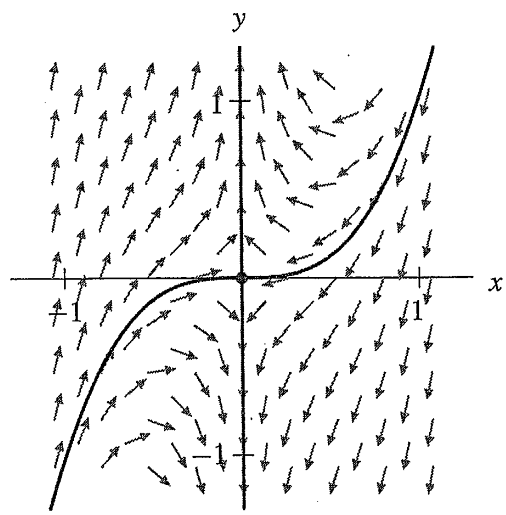
(c) The general solution of the system is

$$x(t) = x_0 e^{-t}$$
$$y(t) = x_0^3 e^{-3t} + (y_0 - x_0^3)e^t.$$

(d) For all solutions, $x(t) \to 0$ as $t \to \infty$. For a solution to tend to the origin as $t \to \infty$, we must have $y(t) \to 0$, and this can happen only if $y_0 - x_0^3 = 0$.

(e) Since $x = x_0 e^{-t}$, we see that a solution will tend toward the origin as $t \to -\infty$ only if $x_0 = 0$. In that case, $y(t) = y_0 e^t$, and $y(t) \to 0$ as $t \to -\infty$.

(f)



(g) Solutions tend away from the origin along the y-axis in both systems. In the nonlinear system, solutions approach the origin along the curve $y = x^3$ which is tangent to the x-axis. For the linearized system, solutions tend to the origin along the x-axis. Near the origin, the phase portraits are almost the same.

(f) The reason the linearizations and the nonlinear system look so different is that the equation for dx/dt contains only higher-order terms (just x^3 in this case). Since the equilibrium points occur along the y-axis ($x = 0$), the linearization has an entire line of equilibria in the x-direction.

18. **(a)** The equation $x^2 - a = 0$ has no solutions if $a < 0$.

(b) The equilibrium points are $(\pm\sqrt{a}, 0)$.

(c) When $a = 0$, the only equilibrium point is $(0, 0)$.

(d) The Jacobian matrix is

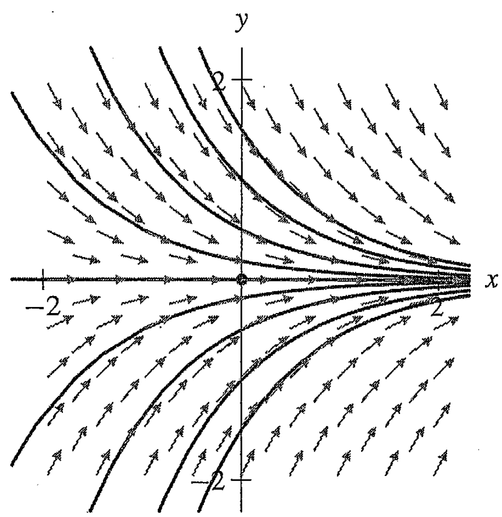
$$\begin{pmatrix} 2x & 0 \\ -2xy & -x^2 - 1 \end{pmatrix}.$$

At $(0, 0)$, the Jacobian matrix is

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},$$

which has eigenvalues -1 and 0. So $(0, 0)$ is a node.

19. **(a)**

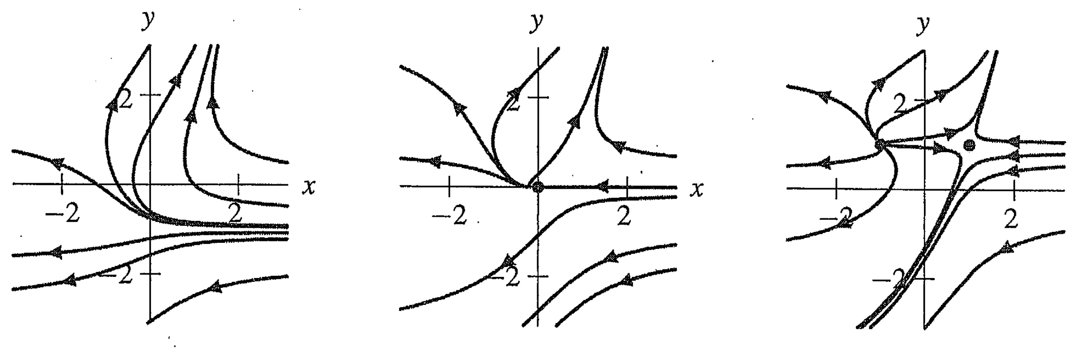


(b) The linearization of the equilibrium point at the origin has the coefficient matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},$$

which has eigenvalues -1 and 0. So for the linearized system, the x-axis is a line of equilibria and solutions tend to zero in the y-direction. The nonlinear terms make solutions tend to zero in the x-direction for initial conditions with $x < 0$ and away from zero in the x-direction for initial conditions with $x > 0$.

(c) Note that for all values of the parameter a, the line $y = a$ is invariant. If $a < 0$, all solutions come from and go to infinity. If $a = 0$, most solutions come from and go to infinity, but there are separatrices associated to the equilibrium point at the origin. If $a > 0$, some solutions come from and go to infinity, but many solutions come from the source at $(-\sqrt{a}, a)$ and go to infinity. There is also a separating solution along the line $y = a$ that comes from the source at $(-\sqrt{a}, a)$ and goes to the saddle at $(\sqrt{a}, a)$.

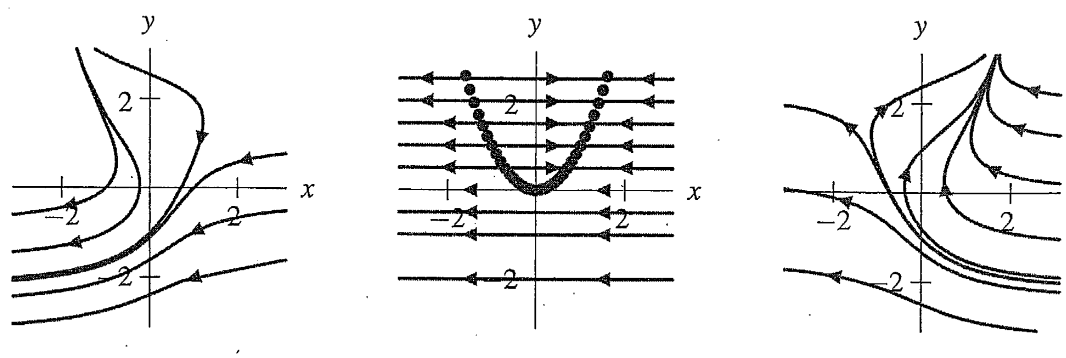


Phase portrait for $a < 0$ Phase portrait for $a = 0$ Phase portrait for $a > 0$

21. **(a)** The only equilibrium points occur if $a = 0$. Then all points on the curve $y = x^2$ are equilibrium points.

(b) The bifurcation occurs at $a = 0$.

(c) If $a < 0$, all solutions decrease in the y-direction since $dy/dt < 0$. If $a > 0$, all solutions increase in the y-direction since $dy/dt > 0$. If $a = 0$, there is a curve of equilibrium points located along $y = x^2$, and all solutions move horizontally.



Phase portrait for $a < 0$ Phase portrait for $a = 0$ Phase portrait for $a > 0$

22. **(a)** The equilibrium points are $((1 \pm \sqrt{1+4a})/2, -(1 \pm \sqrt{1+4a})/2 - a)$, so there are no equilibrium points if $a < -1/4$, one equilibrium if $a = -1/4$, and two equilibrium points if $a > -1/4$.

(b) A bifurcation occurs at $a = -1/4$.

(c) If $a < -1/4$, there are no equilibrium points and all solutions come from and go to infinity. If $a = -1/4$, an equilibrium point appears at $(1/2, 1/4)$. This equilibrium point has both eigenvalues 0 and is a node. If $a > -1/4$, the system has two equilibrium points, at $((1 \pm \sqrt{1+4a})/2, -(1 \pm \sqrt{1+4a})/2 - a)$.