4. The characteristic polynomial is

$$s^2 - 4s + 4,$$

so $s = 2$ is a repeated eigenvalue. Hence, the general solution is

$$y(t) = k_1 e^{2t} + k_2 t e^{2t}.$$

5. The characteristic polynomial is

$$s^2 + 8s + 25,$$

so the complex eigenvalues are $s = -4 \pm 3i$. Hence, the general solution is

$$y(t) = k_1 e^{-4t} \cos 3t + k_2 e^{-4t} \sin 3t.$$

6. The characteristic polynomial is

$$s^2 - 4s + 29,$$

so the complex eigenvalues are $s = 2 \pm 5i$. Hence, the general solution is

$$y(t) = k_1 e^{2t} \cos 5t + k_2 e^{2t} \sin 5t.$$

7. The characteristic polynomial is

$$s^2 + 2s - 3,$$

so the eigenvalues are $s = 1$ and $s = -3$. Hence, the general solution is

$$y(t) = k_1 e^t + k_2 e^{-3t},$$

and we have

$$y'(t) = k_1 e^t - 3k_2 e^{-3t}.$$

From the initial conditions, we obtain the simultaneous equations

$$\begin{cases} k_1 + k_2 = 6 \\ k_1 - 3k_2 = -2. \end{cases}$$

Solving for k_1 and k_2 yields $k_1 = 4$ and $k_2 = 2$. Hence, the solution to our initial-value problem is $y(t) = 4e^t + 2e^{-3t}$.

8. The characteristic polynomial is

$$s^2 + 4s - 5,$$

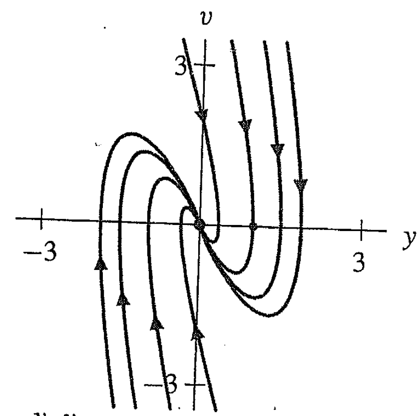
so the eigenvalues are $s = 1$ and $s = -5$. Hence, the general solution is

$$y(t) = k_1 e^t + k_2 e^{-5t},$$

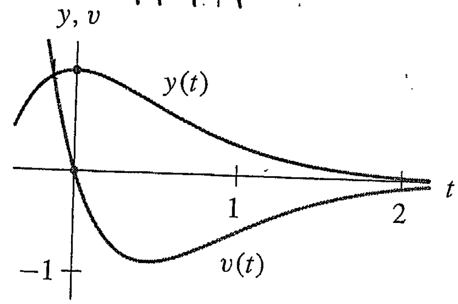
and we have

$$y'(t) = k_1 e^t - 5k_2 e^{-5t}.$$

(d) All solutions tend to the origin spiralling in the clockwise direction with period 2π. Admittedly, it is difficult to see these oscillations in the picture.



(e) The graph of $y(t)$ initially decreases then oscillates with decreasing amplitude as it tends to 0. Similarly, $v(t)$ initially decreases and becomes negative, then oscillates with decreasing amplitude as it tends to 0.



16. **(a)** The resulting second-order equation is

$$\frac{d^2y}{dt^2} + 8y = 0,$$

and the corresponding system is

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -8y. \end{aligned}$$

(b) Recall that we can read off the characteristic equation of the second-order equation straight from the equation without having to revert to the corresponding system. We obtain

$$\lambda^2 + 8 = 0.$$

Therefore, the eigenvalues are $\lambda_1 = 2\sqrt{2}\,i$ and $\lambda_2 = -2\sqrt{2}\,i$.

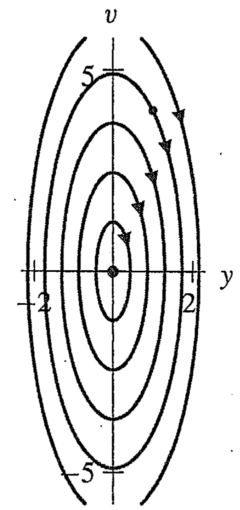
To find the eigenvectors associated to the eigenvalue λ_1, we solve the simultaneous system of equations

$$\begin{cases} v = 2\sqrt{2}\,iy \\ -8y = 2\sqrt{2}\,iv. \end{cases}$$

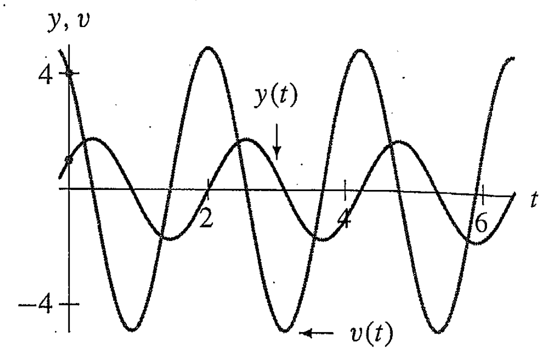
From the first equation, we immediately see that the eigenvectors associated to this eigenvalue must satisfy $v = 2\sqrt{2}\,iy$. Similarly, the eigenvectors associated to the eigenvalue $\lambda_2 = -2\sqrt{2}\,i$ must satisfy the equation $v = -2\sqrt{2}\,iy$.

(c) Since the eigenvalues are pure imaginary the equilibrium point at the origin is a center with natural period $\pi/\sqrt{2}$, and the system is undamped.

(d) All solutions move clockwise.



(e) Each graph is periodic with period $\pi/\sqrt{2}$.



17. **(a)** The resulting second-order equation is

$$2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = 0,$$

and the corresponding system is

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -\frac{1}{2}y - \frac{3}{2}v.$$

(b) Recall that we can read off the characteristic equation of the second-order equation straight from the equation without having to revert to the corresponding system. We obtain

$$2\lambda^2 + 3\lambda + 1 = 0.$$

Therefore, the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -1/2$.

To find the eigenvectors associated to the eigenvalue λ_1, we solve the simultaneous system of equations

$$\begin{cases} v = -y \\ -\frac{1}{2}y - \frac{3}{2}v = -v. \end{cases}$$

From the first equation, we immediately see that the eigenvectors associated to this eigenvalue must satisfy $v = -y$. Similarly, the eigenvectors associated to the eigenvalue $\lambda_2 = -1/2$ must satisfy the equation $v = -y/2$.

(c) Since the eigenvalues are real and negative, the equilibrium point at the origin is a sink, and the system is overdamped.

32. If we let $v = dy/dt$, then the corresponding first-order system is

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -qy - pv,$$

and the corresponding matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}.$$

If λ is an eigenvalue, then it is a root of the characteristic polynomial. In other words,

$$\lambda^2 + p\lambda + q = 0.$$

Now consider

$$\mathbf{A}\begin{pmatrix} 1 \\ \lambda \end{pmatrix} = \begin{pmatrix} \lambda \\ -q - p\lambda \end{pmatrix}$$
$$= \begin{pmatrix} \lambda \\ \lambda^2 \end{pmatrix}$$
$$= \lambda\begin{pmatrix} 1 \\ \lambda \end{pmatrix}.$$

33. **(a)** If we let $v = dy/dt$, then the corresponding first-order system is

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -qy - pv,$$

and the corresponding matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}.$$

If λ_0 is a repeated eigenvalue, then the characteristic polynomial is

$$\lambda^2 + p\lambda + q = (\lambda - \lambda_0^2) = \lambda^2 - 2\lambda_0\lambda + \lambda_0^2.$$

Consequently, $p = -2\lambda_0$, $q = \lambda_0^2$, and

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\lambda_0^2 & 2\lambda_0 \end{pmatrix}.$$

(b) To compute the general solution of the corresponding first-order system, we consider an arbitrary initial condition $\mathbf{V}_0 = (y_0, v_0)$ and calculate

$$(\mathbf{A} - \lambda_0 \mathbf{I})\mathbf{V}_0 = \begin{pmatrix} -\lambda_0 & 1 \\ -\lambda_0^2 & \lambda_0 \end{pmatrix} \begin{pmatrix} y_0 \\ v_0 \end{pmatrix}$$

$$= \begin{pmatrix} -\lambda_0 y_0 + v_0 \\ -\lambda_0^2 y_0 + \lambda_0 v_0 \end{pmatrix}.$$

The general solution of the first-order system is

$$\mathbf{Y}(t) = e^{\lambda_0 t} \begin{pmatrix} y_0 \\ v_0 \end{pmatrix} + t e^{\lambda_0 t} \begin{pmatrix} -\lambda_0 y_0 + v_0 \\ -\lambda_0^2 y_0 + \lambda_0 v_0 \end{pmatrix}.$$

(c) From the first component of the result in part (b), we obtain the general solution of the original second-order equation in the form

$$y(t) = y_0 e^{\lambda_0 t} + (-\lambda_0 y_0 + v_0) t e^{\lambda_0 t}.$$

(d) Let $k_1 = y_0$ and $k_2 = -\lambda_0 y_0 + v_0$. Clearly, all k_1 are possible. Moreover, once the value of k_1 is determined, k_2 can be determined from v_0 using $k_2 = -\lambda_0 k_1 + v_0$, and v_0 can be determined by k_2 using $v_0 = k_2 + \lambda_0 k_1$. Hence, k_1 and k_2 are arbitrary constants because y_0 and v_0 are arbitrary.

34. We must first find out how fast the "typical" solution of this equation approaches the origin.

The characteristic equation for this harmonic oscillator is

$$s^2 + bs + 3 = 0,$$

and the roots are

$$\frac{-b \pm \sqrt{b^2 - 12}}{2}.$$

These roots are complex if $b^2 < 12$, and all solutions tend to the equilibrium at the rate of $e^{(-b/2)t}$.

If $b^2 > 12$, the roots are real, and the general solution is

$$y(t) = k_1 e^{((-b+\sqrt{b^2-12})/2)t} + k_2 e^{((-b-\sqrt{b^2-12})/2)t}.$$

For the typical solution, both k_1 and k_2 are nonzero, so the typical solution tends to the origin at a rate determined by the slower of these two exponentials. The second of these exponential terms tends to 0 most quickly since $-b - \sqrt{b^2 - 12} < -b + \sqrt{b^2 - 12} < 0$. So the typical solution tends to 0 at the rate determined by the exponential of the form $e^{((-b+\sqrt{b^2-12})/2)t}$.

We must determine which of the two exponentials

$$e^{(-b/2)t}$$

(for $b < 2\sqrt{3}$) and

$$e^{((-b+\sqrt{b^2-12})/2)t}$$

which can be written as

$$m\frac{d^2y}{dt^2} - b_{mf}\frac{dy}{dt} + ky = 0.$$

(b) The equivalent first-order system is

$$\begin{aligned}\frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -\frac{k}{m}y + \frac{b_{mf}}{m}v.\end{aligned}$$

(c) The characteristic equation is

$$m\lambda^2 - b_{mf}\lambda + k = 0,$$

and the eigenvalues are

$$\frac{b_{mf} \pm \sqrt{b_{mf}^2 - 4mk}}{2m}.$$

Since m, b_{mf}, and k are all positive parameters, the eigenvalues are either positive real numbers or complex numbers with a positive real part. If both eigenvalues are real, then the origin is called an "overstimulated" source. The magnitudes of $y(t)$ and $v(t)$ tend to infinity without oscillation. If the eigenvalues are complex, then the origin is a spiral source and the oscillator is called understimulated. The solutions spiral away from the origin with natural period $4m\pi/\sqrt{b_{mf}^2 - 4mk}$.

38. We have the second-order differential equation

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0.$$

The characteristic polynomial is

$$m\lambda^2 + b\lambda + k,$$

and the eigenvalues are

$$\frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

In our case, $b^2 - 4mk < 0$, so the eigenvalues can be written as

$$\frac{-b \pm i\sqrt{4mk - b^2}}{2m}.$$

Using this expression for the eigenvalues, we obtain the natural period P as

$$P = \frac{2\pi}{\frac{\sqrt{4mk - b^2}}{2m}} = \frac{4m\pi}{\sqrt{4mk - b^2}}.$$

(a) If $m = 1$, $k = 2$, and $b = 1$, we have a natural period of $4\pi/\sqrt{7}$.

(b) To see how the period changes as m changes, we compute

$$\frac{\partial P}{\partial m} = 4\pi(4mk - b^2)^{-3/2}(2mk - b^2).$$

In our case, $m = 1$, $k = 2$, and $b = 1$. Hence, we have $\partial P/\partial m = 12\pi/(7\sqrt{7})$, and the period increases as the mass increases. The speed that it increases is given by the value of $\partial P/\partial m$, which is $12\pi/(7\sqrt{7})$.

(c) To see how the period changes as k changes, we compute

$$\frac{\partial P}{\partial k} = -8\pi m^2(4mk - b^2)^{-3/2}.$$

In our case, $m = 1$, $k = 2$, and $b = 1$. Hence, we have $\partial P/\partial k = -8\pi/(7\sqrt{7})$, and the period decreases as the spring constant increases. The speed that it increases is given by the value of $\partial P/\partial k$, which is $-8\pi/(7\sqrt{7})$.

(d) To see how the period changes as b changes, we compute

$$\frac{\partial P}{\partial b} = 4\pi mb(4mk - b^2)^{-3/2}.$$

In our case, $m = 1$, $k = 2$, and $b = 1$. Hence, we have $\partial P/\partial b = 4\pi/(7\sqrt{7})$, and the period increases as the damping increases. The speed that it increases is given by the value of $\partial P/\partial b$, which is $4\pi/(7\sqrt{7})$.

39. The differential equation is

$$m\frac{d^2y}{dt^2} + 2y = 0,$$

and the characteristic equation is $m\lambda^2 + 2 = 0$. Hence, the eigenvalues are $\lambda = \pm i\sqrt{2/m}$. The natural period is $2\pi/\sqrt{2/m} = \pi\sqrt{2m}$. For natural period to be 1, we must have $m = 1/(2\pi^2)$.

40. We have the second-order differential equation

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0.$$

The characteristic polynomial is $m\lambda^2 + b\lambda + k$, and the eigenvalues are $(-b \pm \sqrt{b^2 - 4mk})/(2m)$. In our case, $b^2 - 4mk < 0$, so the eigenvalues can be written as

$$\frac{-b \pm i\sqrt{4mk - b^2}}{2m}.$$

Using this expression for the eigenvalues, we obtain the natural period P as

$$P = \frac{2\pi}{\dfrac{\sqrt{4mk - b^2}}{2m}} = \frac{4\pi m}{\sqrt{4mk - b^2}}.$$

Each tick of the clock takes one-half of the period. Consequently, if the period gets longer, the time between ticks gets longer and the clock runs slow. Note that the period is inversely proportional to the quantity $\gamma = \sqrt{4mk - b^2}$.

(a) If b increases slightly, then γ decreases slightly. Hence, the period P increases slightly, and the clock runs slow.